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Onset of convection in a porous mantle

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Abstract. The problem of thermal instability in a fluid saturated porous spherical shell heated internally, due to uniform internal heat sources and in equilibrium under its own radial gravitational field is studied theoretically. A general disturbance is analysed into modes in terms of spherical harmonics of various orders, l, for different values of the thickness of the mantle and the criteria for the onset of convection for the first fifteen modes is obtained in four different cases when the outer and inner bounding surfaces are either impermeable or permeable. It is shown that as the thickness of the shell decreases, the pattern of convection which sets in at marginal stability shifts progressively to harmonics of higher order for all the three cases except when both the bounding surfaces are permeable, in which case the onset of convection occurs at a harmonic of order 1. A comparison of some representative results of these cases is made with that of continuous fluid shell with rigid or free boundary surfaces. The neutral stability plots for various thickness of the mantle, for five different models of the mantle, are plotted for the different types of boundary surfaces.

Keywords. Porous mantle; mantle convection; Darcy–Rayleigh number; Darcy–Prandtl number; geophysical model.

1. Introduction

The problem of convection in a spherical shell heated from within has attracted widespread attention among geophysicists and astrophysicists interested in convective process. Heiskanen and Vening Meinesz [9] suggested that sometime in the early history of the earth cooling, great current system existed in the mantle, chiefly due to temperature gradients caused by cooling at the surface. According to the presently available picture of the earth, its central core about 7,000 km diameter, is composed of solid nickel and iron at temperature of nearly 5000°C and at pressure as high as 3.8 million kg/scm. Wrapped around this core is the mantle, a 3000 km thick area of molten and semisolid rocks and around this is the crust averaging 50 km of thickness and holding the oceans and continents. As a result, it has been suggested by Simpkins and Blythe [14] that thermal convection problems in porous media have application to convection in the mantle of the earth. The problem of mantle convection is discussed here.

and the fluid is treated as a unique fictitious fluid of thermal conductivity k_m and heat capacity $(\rho_c)_m = \hat{\varepsilon}(\rho_c)_f + (1-\hat{\varepsilon})(\rho_c)_s$. This hypothesis has been extensively studied by Combarnous and Bories [7] and [8] for the case of a horizontal porous layer. In the steady state, the temperature distribution $T(r)$ inside the spherical shell is given by

$$\nabla T = -2\beta(r)\mathbf{r} \tag{1}$$

where $\mathbf{r}$ is the position vector with respect to the centre of the sphere,

$$\beta(r) = \beta_2 + (\beta_1/2r^3), \quad \beta_2 = -(H_m/6k_r\kappa_f) \quad \text{and } \beta_1 \text{ is a constant.}$$

The equation of continuity, Darcy equation of motion and energy for a differential volume element of the fluid-porous medium are given by

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

$$(\partial\mathbf{u}/\partial t) + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla(\delta p/\rho_m) + \gamma(r)\Theta\mathbf{r} - (\hat{\varepsilon}\nu/K)\mathbf{u} \tag{3}$$

$$C(\partial\Theta/\partial t) + \hat{\varepsilon}\mathbf{u}\cdot\nabla\Theta = k_r\kappa_f\nabla^2\Theta + 2\hat{\varepsilon}\beta(r)\mathbf{u}\cdot\mathbf{r}. \tag{4}$$

The momentum equation (3) is similar to that used by Lapwood [11]. Here $\mathbf{u}$ is the disturbance velocity, δp the perturbation in pressure, C is the ratio of heat capacities $(\rho_c)_m/(\rho_c)_f$, k_r the thermal conductivity ratio k_m/k_f, k_m being the thermal conductivity of the solid-fluid mixture given by $\hat{\varepsilon}k_f + (1-\hat{\varepsilon})k_s$, f denoting the fluid value, s the solid value, m the solid-fluid mixture and κ_f the thermal diffusivity of the fluid. In deriving the equation of motion and energy, we have assumed the Darcy–Oberbeck–Boussinesq approximation and have replaced ρ_f occurring with the gravity term $g(r)$ by $\rho_f = \rho_m(1-\alpha\Theta)$ and ρ_f occurring elsewhere as a constant equal to ρ_m, the density of the saturated medium; the perturbation density $\delta\rho_f$ and that in the temperature Θ being related by $\delta\rho_f = -\rho_f\alpha\Theta$ and $\gamma(r) = \alpha g(r)$.

2.1. *Eigenvalue problem for spherical shell*

To investigate the necessary criteria for the onset of thermal instability, we need only consider infinitesimal disturbances.

Following Chandrasekhar [5], we apply the operator $\mathbf{r}\cdot\nabla X\nabla X(\text{———})$ to (3) and write the result in spherical polar coordinates (r, θ, ϕ):

$$\left(\frac{\hat{\varepsilon}\nu}{K} + \frac{\partial}{\partial t}\right)\nabla^2(\mathbf{u}\cdot\mathbf{r}) = -\gamma(r)L^2\Theta \tag{5}$$

where

$$L^2 = r^2\left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \nabla^2\right\}$$

$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{6}$$

and ∇^2 is usual Laplacian in spherical polar form. Restricting our attention to the linear problem describing small amplitude convection, we neglect the non-linear terms $\mathbf{u}\cdot\nabla u$ and $\mathbf{u}\cdot\nabla\Theta$ occurring in (3) and (4). The dependent variables $\mathbf{u}\cdot\mathbf{r}$ and Θ occurring in (4) and (5) are now expanded in a complete set of spherical harmonics

$$\mathbf{u}\cdot\mathbf{r} = W(r)\,Y_l^m(\theta,\phi)\exp(pt)$$

$$\Theta = \Theta(r)\,Y_l^m(\theta,\phi)\exp(pt) \tag{7}$$

where

$$Y_l^m(\theta,\phi) = P_l^m(\cos\theta)\exp(\pm \mathrm{Im}\,\phi) \tag{8}$$

and $P_l^m(\cos\theta)$ are the associated Legendre polynomials, p being a constant. Substituting (7) and (8) in (4) and (5) with $m=0$ (for axisymmetric disturbances) yield

$$(1+(K/\hat{\varepsilon}v)p)\mathscr{D}_1 W(r) = -\,l(l+1)(K/\hat{\varepsilon}v)\gamma(r)\Theta(r) \tag{9}$$

and

$$(k_r\kappa_f\mathscr{D}_1 - pC)\Theta(r) = -2\hat{\varepsilon}\beta(r)W(r) \tag{10}$$

where

$$\mathscr{D}_1 = (\mathrm{d}^2/\mathrm{d}r^2) + (2/r)(\mathrm{d}/\mathrm{d}r) - \frac{l(l+1)}{r^2}. \tag{11}$$

We now measure r in units of a suitable radius R_1, the outer radius of the shell and denote $\sigma = (K/\hat{\varepsilon}v)p$ and rewrite (9) and (10) as

$$(1+\sigma)\mathscr{D}_1 W = -l(l+1)\gamma(r)\frac{K}{\hat{\varepsilon}v}R_1^2\Theta \tag{12}$$

and

$$(\mathscr{D}_1 - BC\sigma)\Theta = -2\beta(r)\frac{\hat{\varepsilon}}{k_r\kappa_f}R_1^2 W \tag{13}$$

where the Darcy–Prandtl number B^{-1} is defined by

$$B = \frac{\hat{\varepsilon}vR_1^2}{Kk_r\kappa_f}. \tag{14}$$

2.2. *Boundary conditions*

We use spherical polar coordinates (r,θ,ϕ) with origin at the centre of the core and θ as colatitude and confine the shell between radii $r=\eta$ and $r=1$, the unit of length being chosen as the outer radius R_1 of the shell. The spherical boundaries are assumed to be either impermeable or else they are bounded by standing fluid identical with that saturating the medium. In both the cases the boundary conditions are

$$\Theta = 0 \quad \text{for } r=1 \text{ and } r=\eta. \tag{15}$$

Further, on the impermeable boundaries, we have

$$W = 0 \quad \text{for } r=1 \text{ and } r=\eta \tag{16}$$

Considering the implication of imposing condition (17) on spherical boundaries from the θ- and ϕ-components of equation of motion, we deduce on making use of the incompressibility condition (2) that

$$(\mathrm{d}W/\mathrm{d}r) + (W/r) = 0 \quad \text{for } r = 1 \text{ and } r = \eta \tag{18}$$

for permeable boundaries.

2.3. *Exchange of stabilities for the case β = constant and γ = constant*

With the restriction $\beta(r) = \beta$ and $\gamma(r) = \gamma$ and substituting

$$F = -l(l+1)\gamma \frac{K}{\hat{\varepsilon}\nu} R_1^2 \Theta, \tag{19}$$

(12) and (13) reduce to

$$(1+\sigma)\mathscr{D}_l W = F \tag{20}$$

and

$$(\mathscr{D}_l - BC\sigma)F = l(l+1)R_D W \tag{21}$$

where

$$R_D = \frac{2\beta\gamma K}{k_r \kappa_f \nu} R_1^4. \tag{22}$$

The modified boundary conditions are now

$$F = 0 \quad \text{for } r = 1 \text{ and } r = \eta \tag{23}$$

$W = 0$ for $r = 1$ and $r = \eta$ on impermeable boundaries and

$$(\mathrm{d}W/\mathrm{d}r) + (W/r) = 0 \quad \text{for } r = 1 \text{ and } r = \eta \tag{24}$$

on permeable boundaries.

We now multiply (21) by $r^2 F^*$, superscript * denoting complex conjugate of the quantity, integrate the result over the range $(\eta, 1)$ and utilise (20) to obtain

$$\int_\eta^1 \left\{ r^2 \left| \frac{\mathrm{d}F}{\mathrm{d}r} \right|^2 + l(l+1)|F|^2 + BC\sigma r^2 |F|^2 \right\} \mathrm{d}r$$

$$+ l(l+1)R_D(1+\sigma^*) \left[r^2 W \frac{\mathrm{d}W^*}{\mathrm{d}r} \Big|_\eta^1 - \int_\eta^1 \left\{ r^2 \left| \frac{\mathrm{d}W}{\mathrm{d}r} \right|^2 + l(l+1)|W|^2 \right\} \mathrm{d}r \right] = 0. \tag{25}$$

In deriving (25) we have made use of the property of the operator $\mathscr{D}_l$

$$\int_\eta^1 r^2 \phi \mathscr{D}_l \psi \, \mathrm{d}r = r^2 \phi \frac{\mathrm{d}\psi}{\mathrm{d}r} \Big|_\eta^1 - \int_\eta^1 \left\{ r^2 \frac{\mathrm{d}\psi}{\mathrm{d}r} \frac{\mathrm{d}\phi}{\mathrm{d}r} + l(l+1)\phi\psi \right\} \mathrm{d}r, \tag{26}$$

where $\phi(r)$ and $\psi(r)$ are any two bounded and continuous functions in $(\eta, 1)$ and the

On the impermeable boundary $W=0$, the integrated part $[r^2 W(\mathrm{d}W^*/\mathrm{d}r)]_\eta^1$ occurring in (25) vanishes but if the boundary is permeable, then $(\mathrm{d}W^*/\mathrm{d}r)+(W^*/r)=0$, so that

$$\left[r^2 W\frac{\mathrm{d}W^*}{\mathrm{d}r}\right]_\eta^1 = [-r|W|^2]_\eta^1.$$

Hence in either case the integrated part can be written in the form

$$[-r|W|^2]_\eta^1. \tag{27}$$

Vanishing of the imaginary part of (25) with the aid of (27) gives

$$\mathrm{Im}(\sigma)\left[BC\int_\eta^1 r^2|F|^2\,\mathrm{d}r + l(l+1)R_D\left\{\int_\eta^1\left(r^2\left|\frac{\mathrm{d}W}{\mathrm{d}r}\right|^2 + l(l+1)|W|^2\right)\mathrm{d}r\right\} + [r|W|^2]_\eta^1\right]=0. \tag{28}$$

For impermeable boundaries, the factor of $\mathrm{Im}(\sigma)$ in (28) is positive definite, and accordingly

$$\mathrm{Im}(\sigma)=0 \tag{29}$$

for impermeable boundaries.

2.4. *Variational principle when β and γ are constants*

On the assumption of validity of P.E.S. when β and γ are constants, the equations governing the marginal state are given by

$$\mathscr{D}_l W = F \tag{30}$$

and

$$\mathscr{D}_l F = l(l+1)R_D W. \tag{31}$$

These two equations together with boundary conditions (23) and (24) constitute a characteristic value problem for R_D and the instability will set in with that value of l which leads to the lowest value of R_D. The underlying characteristic value problem can be formulated in terms of a variational principle, the basis of which follows from (25), which, when $\sigma=0$ gives

$$l(l+1)R_D = \frac{\displaystyle\int_\eta^1\left\{r^2\left(\frac{\mathrm{d}F}{\mathrm{d}r}\right)^2 + l(l+1)F^2\right\}\mathrm{d}r}{\displaystyle[rW^2]_\eta^1 + \int_\eta^1\left\{r^2\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 + l(l+1)W^2\right\}\mathrm{d}r}. \tag{32}$$

It can be shown as in Chandrasekhar [5], that the lowest characteristic value for R_D is the absolute minimum of the quantity on the right hand side of (32).

$$\mathscr{D}_l\Theta = -2\beta(r)\frac{\hat{\varepsilon}}{k_r\kappa_f}R_1^2 W \tag{34}$$

where R_1 is the radius of outer spherical boundary. With the substitution

$$\beta(r) = \beta_1 b(r), \quad \gamma(r) = \gamma_1 c(r) \tag{35}$$

and

$$F = -l(l+1)\gamma_1 c(r)\frac{K}{\hat{\varepsilon}\nu}R_1^2\Theta \tag{36}$$

where β_1 and γ_1 are the values of $\beta(r)$ and $\gamma(r)$ at $r = 1$, (33) and (34) are rewritten in the form

$$\mathscr{D}_l W = F \tag{37}$$

and

$$\frac{1}{b(r)}\mathscr{D}_l\left\{\frac{F}{c(r)}\right\} = l(l+1)R_D W \tag{38}$$

where the dimensionless parameter occurring in (38) is the Darcy–Rayleigh number

$$R_D = \frac{2\beta_1\gamma_1 K}{k_r\kappa_f\nu}R_1^4 \tag{39}$$

which is defined in terms of the strength of the energy source H_m, the only means by which destabilising density differences are produced in the fluid. Solution of (37) and (38) must be sought which satisfy the boundary conditions (23) and (24).

To solve the characteristic value problem by the variational method and determine the onset of instability as a spherical harmonic of order l, we expand F in a series of cylinder functions of order $l+\frac{1}{2}$ which vanish at $r = 1$ and $r = \eta$. This expansion is possible because of the cylindrical symmetry ($m = 0$) of the problem.

Let

$$C_{l+1/2,\nu}(z) = J_{-(l+1/2)}(\alpha\eta)J_\nu(z) - J_{l+1/2}(\alpha\eta)J_{-\nu}(z) \tag{40}$$

where α is a constant not specified for the present.

Now

$$C_{l+1/2,l+1/2}(\alpha r) = J_{-(l+1/2)}(\alpha\eta)J_{l+1/2}(\alpha r) - J_{l+1/2}(\alpha\eta)J_{-(l+1/2)}(\alpha r) \tag{41}$$

clearly vanishes for $r = \eta$ and it will also vanish for $r = 1$, provided

$$J_{-(l+1/2)}(\alpha\eta)J_{l+1/2}(\alpha) - J_{l+1/2}(\alpha\eta)J_{-(l+1/2)}(\alpha) = 0. \tag{42}$$

We now express F as a series in the form

$$F = \frac{1}{r^{1/2}}\sum_j A_j C_{l+1/2,l+1/2}(\alpha_j r) \tag{43}$$

$$\mathscr{D}_l W_j = \frac{1}{r^{1/2}} C_{l+1/2,l+1/2}(\alpha_j r) \tag{45}$$

which satisfies the necessary boundary conditions at $r = 1$ and $r = \eta$. The general solution of (45) is given by

$$W_j = -\frac{1}{\alpha_j^2 r^{1/2}} C_{l+1/2,l+1/2}(\alpha_j r) + B_1^{(j)} r^1 + B_2^{(j)} r^{-(l+1)} \tag{46}$$

where $B_1^{(j)}$ and $B_2^{(j)}$ are constants of integration to be determined by the boundary conditions. Substitution of solutions for F and W in accordance with (43), (44) and (46) into (38), we get

$$\sum_j A_j \frac{1}{b(r)} \mathscr{D}_l \left\{ \frac{C_{l+1/2,l+1/2}(\alpha_j r)}{c(r) r^{1/2}} \right\}$$

$$= l(l+1) R_D \sum_j A_j \left\{ \frac{-1}{\alpha_j^2 r^{1/2}} C_{l+1/2,l+1/2}(\alpha_j r) + B_1^{(j)} r^1 + B_2^{(j)} r^{-(l+1)} \right\}. \tag{47}$$

We now multiply (47) by $r^{3/2}\, C_{l+1/2,l+1/2}(\alpha_k r)$ and integrate the result over the range of r from η to 1 and obtain

$$\sum_j \left\{ P_{k,j}^{b,c} + l(l+1) R_D \left[-\frac{N_{l+1/2,k}}{\alpha_k^2} \delta_{kj} + Q_{kj} \right] \right\} A_j = 0 \tag{48}$$

where

$$P_{k,j}^{b,c} = -\int_\eta^1 \left\{ \frac{r^2 C_{l+1/2,l+1/2}(\alpha_k r)}{b(r) r^{1/2}} \mathscr{D}_l \frac{C_{l+1/2,l+1/2}(\alpha_j r)}{c(r) r^{1/2}} \right\} \mathrm{d}r \tag{49}$$

and

$$Q_{kj} = \int_\eta^1 C_{l+1/2,l+1/2}(\alpha_k r)(B_1^{(j)} r^{l+3/2} + B_2^{(j)} r^{-l+1/2}) \mathrm{d}r. \tag{50}$$

Equation (48) now leads to the secular equation

$$\left\| P_{k,j}^{b,c} + l(l+1) R_D \left[-\frac{N_{l+1/2,k}}{\alpha_k^2} \delta_{kj} + Q_{kj} \right] \right\| = 0. \tag{51}$$

Owing to the Hermitian character of the operator $r^2 \mathscr{D}_l$ with respect to the function which vanish at the end of the intervals, it follows that

$$P_{k,j}^{b,c} = P_{j,k}^{c,b}. \tag{52}$$

The matrix element $Q_{k,j}$ can be evaluated explicitly with the use of recurrence formulae

$$Q_{kj}=\frac{1}{\alpha_k}[C_{l+1/2,l+3/2}(\alpha_k)-\eta^{l+3/2}C_{l+1/2,l+3/2}(\alpha_k\eta)]B_1^{(j)}$$

$$-\frac{1}{\alpha_k}[C_{l+1/2,l-1/2}(\alpha_k)-\eta^{-l+1/2}C_{l+1/2,l-1/2}(\alpha_k\eta)]B_2^{(j)}. \tag{53}$$

Further, the use of recurrence formulae satisfied by the function $C_{l+1/2,\nu}(s)$ reduces the relation (53) to the form

$$Q_{kj}=-\frac{1}{\alpha_k}\{G_{l+1/2,k}B_1^{(j)}+H_{l+1/2,k}B_2^{(j)}\}. \tag{54}$$

Further reduction of Q_{kj} requires an explicit consideration of the nature of bounding surfaces at $r=1$ and $r=\eta$. We now consider four possible cases separately for further reduction of the matrix element Q_{kj}.

Case 1. Impermeable surfaces at $r=1$ and $r=\eta$.

We apply the conditions (24) for impermeable boundaries at $r=1$ and $r=\eta$ to the solution (46) and obtain the values of $B_1^{(j)}$, and $B_2^{(j)}$. On inserting the values of $B_1^{(j)}$ and $B_2^{(j)}$ so obtained in (54) for $Q_{k,j}$, we get

$$Q_{kj}=0. \tag{55}$$

Case 2. Impermeable surface at $r=1$ and permeable surface at $r=\eta$.

Condition (24) for impermeable boundary at $r=1$ and permeable boundary at $r=\eta$ are applied to solution (46) to obtain the values of $B_1^{(j)}$ and $B_2^{(j)}$. The values of $B_1^{(j)}$ and $B_2^{(j)}$ so obtained, are then inserted into (54) to give

$$Q_{kj}=-\frac{C'_{l+1/2}(\alpha_j\eta)}{\alpha_k\alpha_j\{(l+1)\eta^{l+1/2}+l\eta^{-(l+3/2)}\}}[G_{l+1/2,k}-H_{l+1/2,k}] \tag{56}$$

which is symmetric in k and j.

Case 3. Permeable surface at $r=1$ and impermeable surface at $r=\eta$.

In a similar manner, we insert the values of $B_1^{(j)}$ and $B_2^{(j)}$ obtained on applying the relevant conditions from (24) to the solution (46), into (54) and reduce the value of Q_{kj} to

$$Q_{kj}=-\frac{C'_{l+1/2}(\alpha_j)}{\alpha_k\alpha_j\{l\eta^{(2l+1)}+(l+1)\}}\{G_{l+1/2,k}-\eta^{(2l+1)}H_{l+1/2,k}\}. \tag{57}$$

We observe that the value of Q_{kj} is symmetric in k and j.

Case 4. Permeable surfaces at $r=1$ and $r=\eta$.

We apply the boundary conditions (24) for permeable boundaries to solution (46) and then insert the values of $B_1^{(j)}$ and $B_2^{(j)}$ so obtained into relation (54) and obtain

$$Q_{kj}=-\frac{\{lG_{l+1/2,j}G_{l+1/2,k}+(l+1)\eta^{(2l+1)}H_{l+1/2,j}H_{l+1/2,k}\}}{\alpha_j\alpha_k\{l(l+1)(1-\eta^{(2l+1)})\}} \tag{58}$$

problem is seen to be self-adjoint and consequently, solving the secular equation (51) is equivalent to A_j's as variational parameters and minimizing the expression (32) for R_D.

4. Geophysical models

We now consider a few models of geophysical importance.

4.1. *Model of the mantle when both temperature distribution and gravity distribution are linear*

The fluid is subject to a spherically symmetric radial gravitational field $g(r)\mathbf{r}$, where $g(r)$ is a function of r only. Hence for a spherical shell of density ρ_m overlying a core of radius R_i and mass M_i,

$$g(r) = \mathscr{G}\left\{(M_i - \tfrac{3}{4}\pi\rho_m R_i^3)\frac{1}{r^3} + \tfrac{4}{3}\pi\rho_m\right\} \tag{59}$$

where $\mathscr{G}$ is the constant of gravitation.

For such a model the distribution of heat sources in the shell is uniform and the gravity varies as r and as such

$$b(r) = c(r) = 1. \tag{60}$$

Now (49) for specific condition (60) gives

$$P_{k,j}^{1,1} = -\int_\eta^1 r^2 \frac{C_{l+1/2,l+1/2}(\alpha_j r)}{r^{1/2}} \mathscr{D}_1 \frac{C_{l+1/2,l+1/2}(\alpha_k r)}{r^{1/2}}\,dr$$

$$= \alpha_k^2 N_{l+1/2,j}\delta_{kj}. \tag{61}$$

The secular equation (51) with the substitution of the value of $P_{k,j}^{1,1}$ from (61) reduces to

$$\left\| N_{l+1/2,k}\left\{\frac{\alpha_k^2}{l(l+1)R_D} - \frac{1}{\alpha_k^2}\right\}\delta_{kj} + Q_{kj}\right\| = 0. \tag{62}$$

And in the first approximation (62) gives

$$l(l+1)R_D = \frac{N_{l+1/2,1}\alpha_1^4}{N_{l+1/2,1} - Q_{11}\alpha_1^2} \tag{63}$$

where α_1 is a first zero of (42) given in Chandrasekhar and Elbert [6]. The above formula (63) together with the expressions for Q_{kj} obtained in cases 1–4 earlier, has been used to determine the characteristic number R_D for various values of l and η and for the set of four boundary conditions when the gravity varies as the radius.

Table 1. Characteristic number R_D for various values of l and η when gravity varies as r. Impermeable surfaces at $r = 1$ and $r = \eta$.

l	$\eta = 0{\cdot}2$ ($\times 10^2$)	$\eta = 0{\cdot}3$ ($\times 10^2$)	$\eta = 0{\cdot}4$ ($\times 10^2$)	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^3$)
1	2·41172	3·23320	5·05558	9·32745	2·10522†	3·10561‡
2	1·88564*	2·10474	2·72258	4·26398	8·46827	1·07681‡
3	1·99335	2·06019*	2·34118	3·15645	5·44639	5·70407
4	2·24224	2·26159	2·39504	2·88668*	4·37503	3·68690
5	2·55400	2·55922	2·62086	2·92913	3·98466	2·68745
6		2·90953	2·93674	3·12992	3·91102*	2·12608
7			3·30909	3·42796	4·01674	1·78507
8				3·79359	4·23927	1·56793
9				4·21033	4·54633	1·42631
10				4·66804	4·91917	1·33384
11				5·16056	5·34624	1·27513
12				5·68376	5·81948	1·24072
13				6·23549	6·33336	1·22450
14				6·81377	6·88355	1·22246*
15				7·41780	7·46689	1·23175

* Denotes minimum value of R_D; † $\times 10^3$; ‡ $\times 10^4$.

Table 2. Characteristic number R_D for various values of l and η when gravity varies as r. Impermeable surface at $r = 1$ and permeable surface at $r = \eta$.

l	$\eta = 0{\cdot}3$ ($\times 10^2$)	$\eta = 0{\cdot}4$ ($\times 10^2$)	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^2$)
1	2·13882	2·69862	4·17776	8·29264	1·29763†
2	1·86146*	2·03550*	2·59112	4·18020	3·92415‡
3	1·99294	2·06542	2·36535*	3·29216	2·24362‡
4	2·24288	2·27407	2·45463	3·08800*	1·58127‡
5	2·55425	2·56723	2·67700	3·14146	1·26053‡
6	2·90827	2·91347	2·97951	3·32867	1·08814‡
7		3·29931	3·33800	3·60194	9·91470
8			3·74029	3·93869	9·38314
9			4·17925	4·32670	9·12473
10			4·66685	4·75834	9·05127
11			5·15058	5·22871	9·10774
12			5·67826	5·73403	9·27153
13			6·23251	6·27168	9·51255

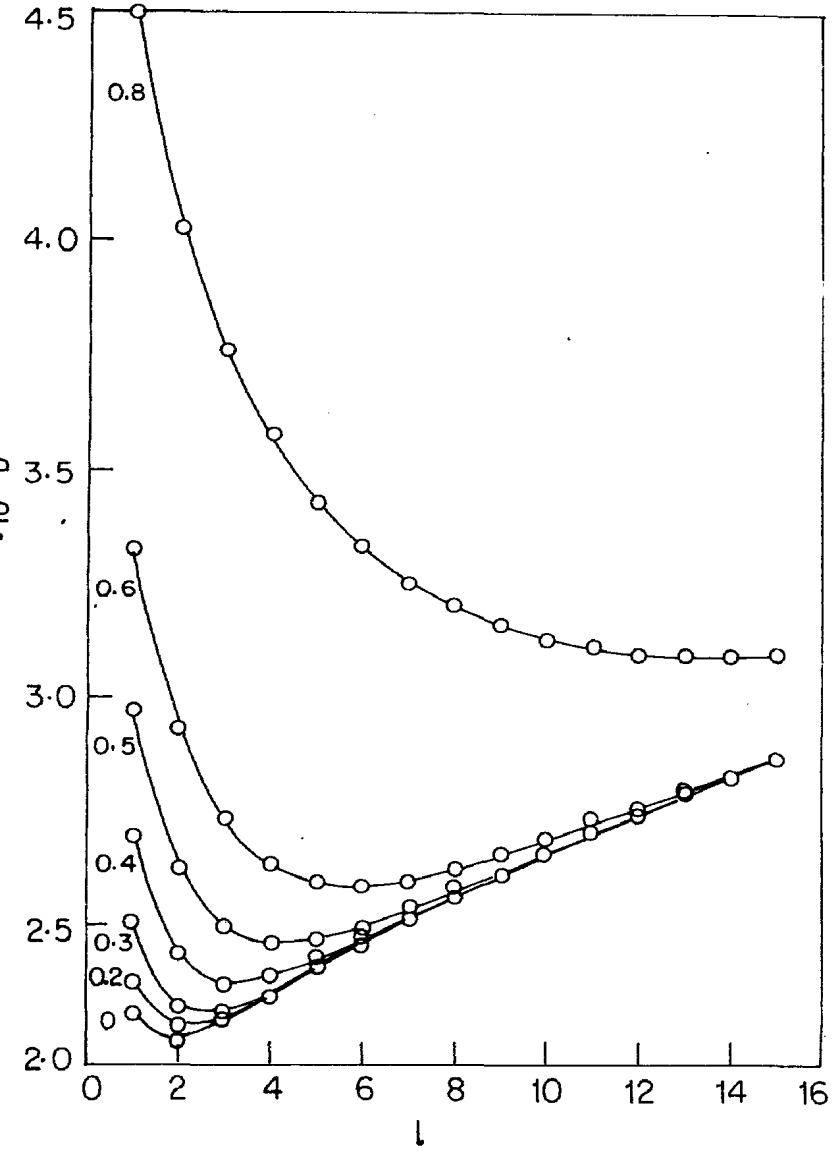
Table 3. Characteristic number R_D for various values of l and η when gravity varies as r. Permeable surface at $r = 1$ and impermeable surface at $r = \eta$.

l	$\eta = 0{\cdot}2$ ($\times 10^2$)	$\eta = 0{\cdot}3$ ($\times 10^2$)	$\eta = 0{\cdot}4$ ($\times 10^2$)	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^2$)
1	1·15110	1·45243	2·12036	3·65983	7·79164	1·06159†
2	1·11932*	1·20183*	1·44640*	2·05567	3·66510	3·85795‡
3	1·32670	1·35084	1·49245	1·80867*	2·75820	2·17674‡
4	1·60126	1·60777	1·66049	1·85971	2·53222*	1·51343‡
5	1·91528	1·91708	1·94035	2·07514	2·56233	1·19156‡
6		2·26231	2·27204	2·35524	2·72536	1·01787‡
7			2·64179	2·69162	2·97413	9·19768
8				3·07201	3·28629	8·65060
9				3·48913	3·64973	8·37564
10				3·93831	4·05692	8·28483*
11				4·41663	4·50298	8·32411
12				4·92212	4·98414	8·46849
13				5·45385	5·49776	8·69044
14				6·01095	6·04160	8·97900
15				6·59244	6·61396	9·32415

* Denotes minimum value of R_D; † $\times 10^4$; ‡ $\times 10^3$.

Table 4. Characteristic number R_D for various values of l and η when gravity varies as r. Permeable surfaces at $r = 1$ and $r = \eta$ where $0{\cdot}5 = \eta \leqslant r \leqslant 1$.

l	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^2$)
1	1·04286*	1·32370*	3·84037*
2	1·17788	1·43330	3·91109
3	1·37402	1·59519	4·01680
4	1·60119	1·80663	4·15736
5	1·92173	2·06454	4·33227
6	2·25938	2·36549	4·54135
7	2·63192	2·70653	4·78374
8	3·03539	3·08466	5·04196
9	3·46714	3·49720	5·36741
10	3·92540	3·94181	5·70752
11	4·40927	4·41662	6·07600
12	4·91796	4·91992	6·48145
13	5·45156	5·45048	6·91417



ure 1. Neutral stability plots for the mantle ($b(r) = 1 = c(r)$) with permeable boundaries both at $r = 1$ and $r = \eta$.

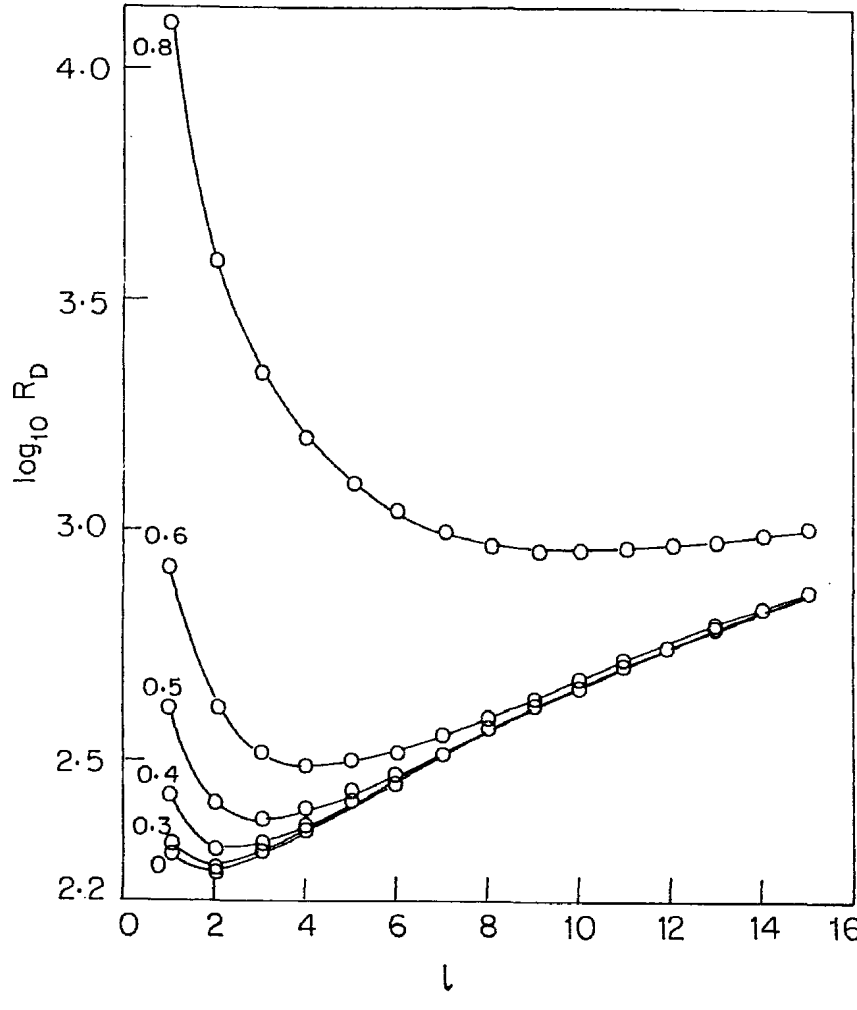


Figure 2. Neutral stability plots for the mantle ($b(r) = 1 = c(r)$) wit impermeable boundary at $r = 1$ and permeable boundary at $r = \eta$.

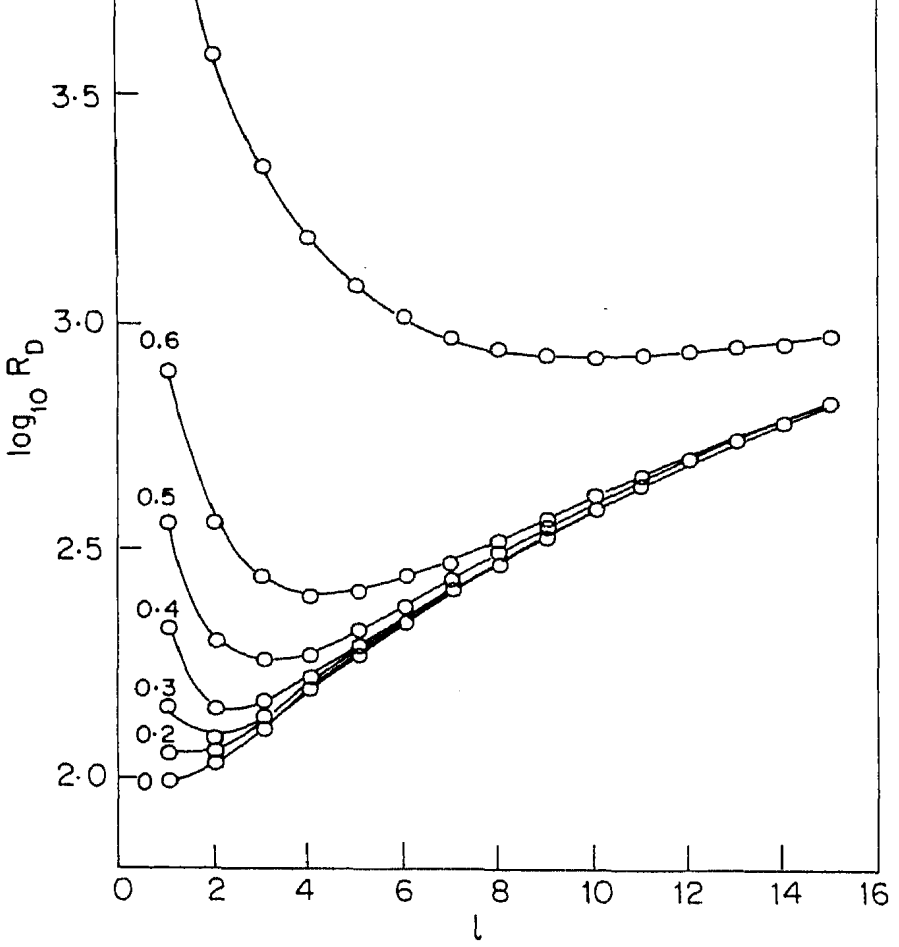


Figure 3. Neutral stability plots for the mantle ($b(r) = 1 = c(r)$) with permeable boundary at $r = 1$ and impermeable boundary at $r = \eta$.

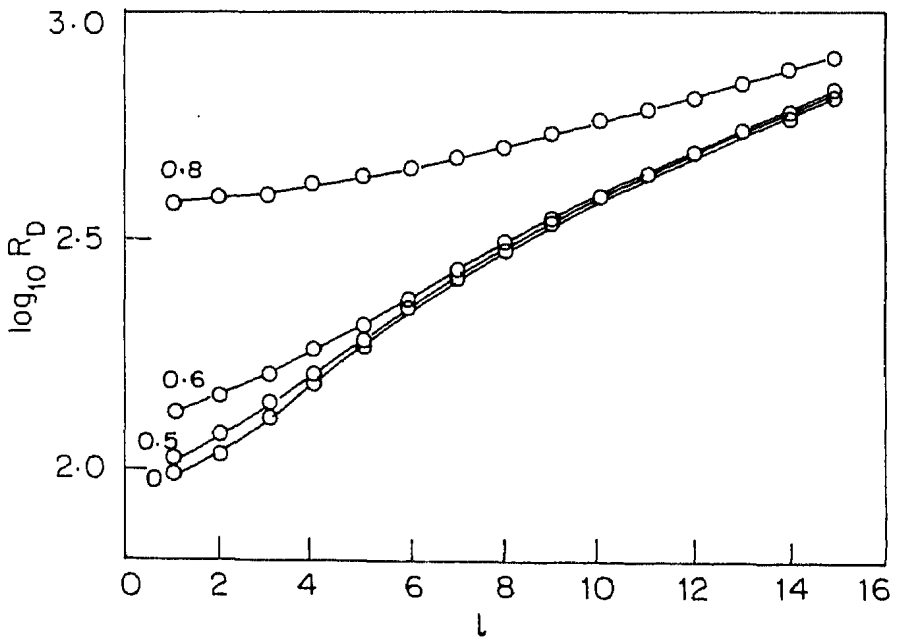


Figure 4. Neutral stability plots for the mantle ($b(r) = 1 = c(r)$) with permeable boundaries both at $r = 1$ and $r = \eta$ ($0{\cdot}5 \leqslant r \leqslant 1$).

As suggested by Chandrasekhar [5] and Lyttkens (refer p. 243 of [5]), in the mantle of the earth, the value of gravity remains approximately constant. Here we consider the case of a model with uniform distribution of heat sources and constant gravity through the mantle, so that we have

$$b(r) = 1 \quad \text{and} \quad c(r) = r^{-1}. \tag{64}$$

For such a specific case, (49) assumes the form

$$P_{k,j}^{1,c(r)} = \alpha_k^2 \int_\eta^1 \{rC_{l+1/2,l+1/2}(\alpha_k r) \cdot rC_{l+1/2,l+1/2}(\alpha_j r)\}\, dr$$

$$= \alpha_k^2 (k|r|j). \tag{65}$$

The characteristic equation (51) with the aid of (65) becomes

$$\left\| \alpha_k^2 (k|r|j) - l(l+1) R_D \left[\frac{N_{l+1/2,j}}{\alpha_k^2} \delta_{kj} - Q_{kj} \right] \right\| = 0 \tag{66}$$

and in the first approximation, (66) gives

$$l(l+1) R_D = \frac{(1|r|1)\alpha_1^4}{N_{l+1/2,1} - Q_{11}\alpha_1^2} \tag{67}$$

This relation (67) together with the expressions for Q_{kj} obtained for the four sets of boundary conditions has been used to determine R_D for various values of r and η, when the gravity is constant through the mantle. The values so obtained have been tabulated in tables 5–8 and have further been illustrated in figures 5–8.

Table 5. Characteristic number R_D for various values of l and η when gravity is constant through the mantle. Impermeable surfaces at $r = 1$ and $r = \eta$.

l	$\eta = 0{\cdot}2$ ($\times 10^2$)	$\eta = 0{\cdot}4$ ($\times 10^2$)	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^3$)
1	1·49768	3·55913	7·00491	1·68417†	2·79505‡
2	1·23321*	1·93575	3·21504	6·78309	9·69131
3	1·36943	1·68799*	2·39259	4·37345	5·13367
4	1·60320	1·75557	2·20254*	3·52190	3·31821
5	1·88485	1·95778	2·25543	3·21562	2·41870
6		2·23192	2·43195	3·16401*	1·91347
7		2·55793	2·68752	3·26160	1·60835
8			3·00452	3·45924	1·41271
9			3·36826	3·72345	1·28510
10			3·77177	4·04848	1·20179
11			4·21102	4·42134	1·14089
12			4·67774	4·83599	1·11913
13			5·16922	5·29469	1·10450*
14			5·68950	5·78218	1·10266*
15			6·23837	6·30205	1·11104

Table 6. Characteristic number R_D for various values of l and η when gravity is constant through the mantle. Impermeable surface at $r = 1$ and permeable surface at $r = \eta$.

l	$\eta = 0{\cdot}2$ ($\times 10^2$)	$\eta = 0{\cdot}4$ ($\times 10^2$)	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^2$)
1	1·23606	1·89983	3·13750	6·63411	1·16786†
2	1·21219*	1·44724*	1·95371	3·34834	3·53174‡
3	1·36488	1·48917	1·79294*	2·64360	2·01925‡
4	1·60261	1·66690	1·87288	1·48580*	1·42314‡
5	1·88504	1·91772	2·06129	2·53516	1·13447‡
6		2·21424	2·31508	2·69289	9·79329
7		2·55037	2·61699	2·92477	8·93314
8			2·96231	3·21397	8·45421
9			3·34340	3·54357	8·22138
10			3·77081	3·91611	8·15519*
11			4·20287	4·32415	8·20607
12			4·67321	4·76498	8·36292
13			5·16675	5·24312	8·58032
14			5·68818	5·74508	8·85809
15			6·23767	6·27567	9·18722

* Denotes minimum value of R_D; † $\times 10^4$; ‡ $\times 10^3$.

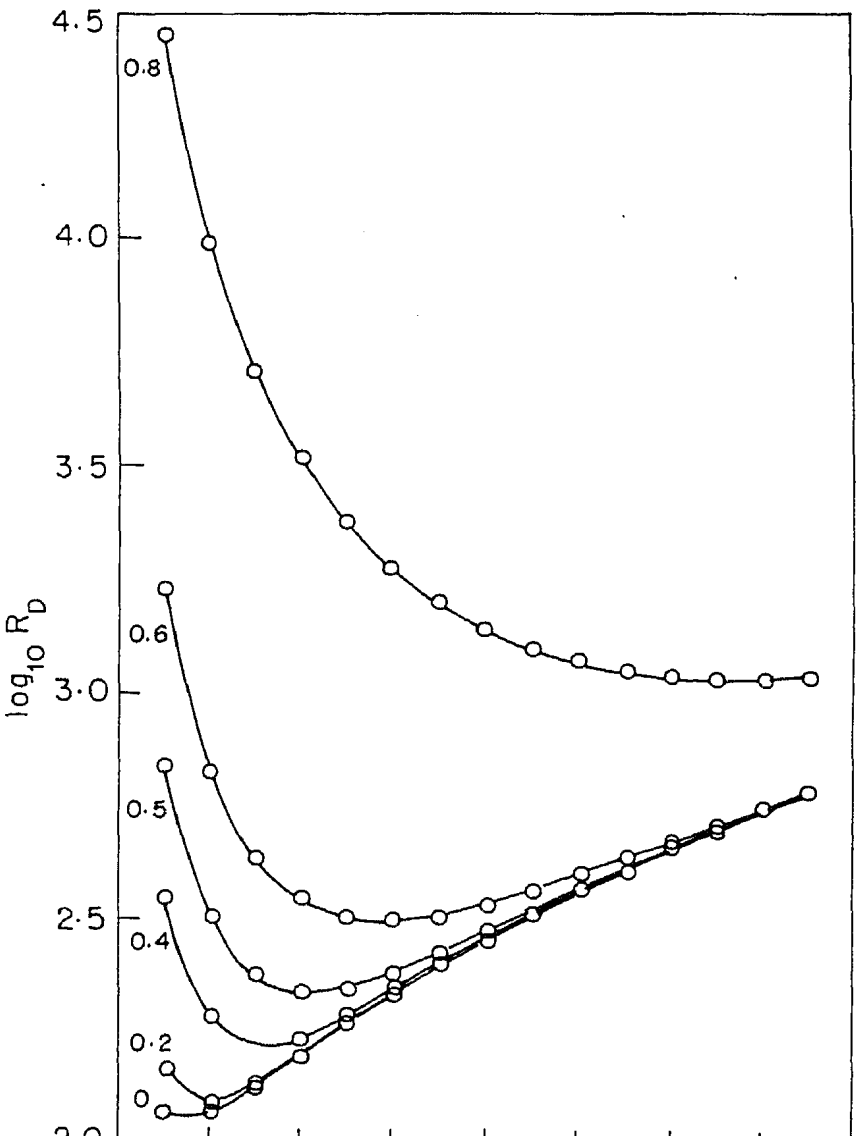
Table 7. Characteristic number R_D for various values of l and η when gravity is constant through the mantle. Permeable surface at $r = 1$ and impermeable surface at $r = \eta$.

l	$\eta = 0{\cdot}2$ ($\times 10^2$)	$\eta = 0{\cdot}4$ ($\times 10^2$)	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^2$)
1	0·71483*	1·49273	2·74853	6·23331	9·55427†
2	0·73204	1·02839*	1·54997	2·93574	3·47216†
3	0·91144	1·07606	1·37097*	1·21483	1·95907†
4	1·14490	1·21714	1·41896	2·03844*	1·36209†
5	1·41348	1·44944	1·59786	2·06780	1·07240†
6		1·72675	1·83002	2·20482	9·16086
7		2·04211	2·11023	2·41499	8·28711
8			2·43303	2·68161	7·79419
9			2·79131	2·98913	7·54645
10			3·18215	3·33885	7·46463
11			3·60397	3·72396	7·50000
12			4·05091	4·14182	7·63858
13			4·52124	4·59612	7·83878

Table 3. Characteristic number R_D for various values of l and η when gravity is constant through the mantle. Permeable surfaces at $r = 1$ and $r = \eta$.

l	$\eta = 0{\cdot}2$ ($\times 10^2$)	$\eta = 0{\cdot}4$ ($\times 10^2$)	$\eta = 0{\cdot}5$ ($\times 10^2$)	$\eta = 0{\cdot}6$ ($\times 10^2$)	$\eta = 0{\cdot}8$ ($\times 10^2$)
1	0·58915*	0·65266*	0·78319*	1·05896*	3·45633*
2	0·71570	0·77255	0·88812	1·14807	3·51998
3	0·90904	0·96335	1·04150	1·28094	3·61512
4	1·44457	1·16564	1·22171	1·45434	3·74162
5	1·41345	1·42499	1·47973	1·66608	3·89904
6		1·71547	1·75554	1·91368	4·08721
7		2·03710	2·06342	2·19770	4·31015
8			2·40403	2·51708	4·54281
9			2·77371	2·86421	4·83603
10			3·17172	3·24411	5·14248
11			3·59797	3·65255	5·47445
12			4·04748	4·08845	5·84627
13			4·51934	4·55660	6·23658
14			5·01811	5·04602	6·65417
15			5·54369	5·56139	7·09829

* Denotes minimum value of R_D.



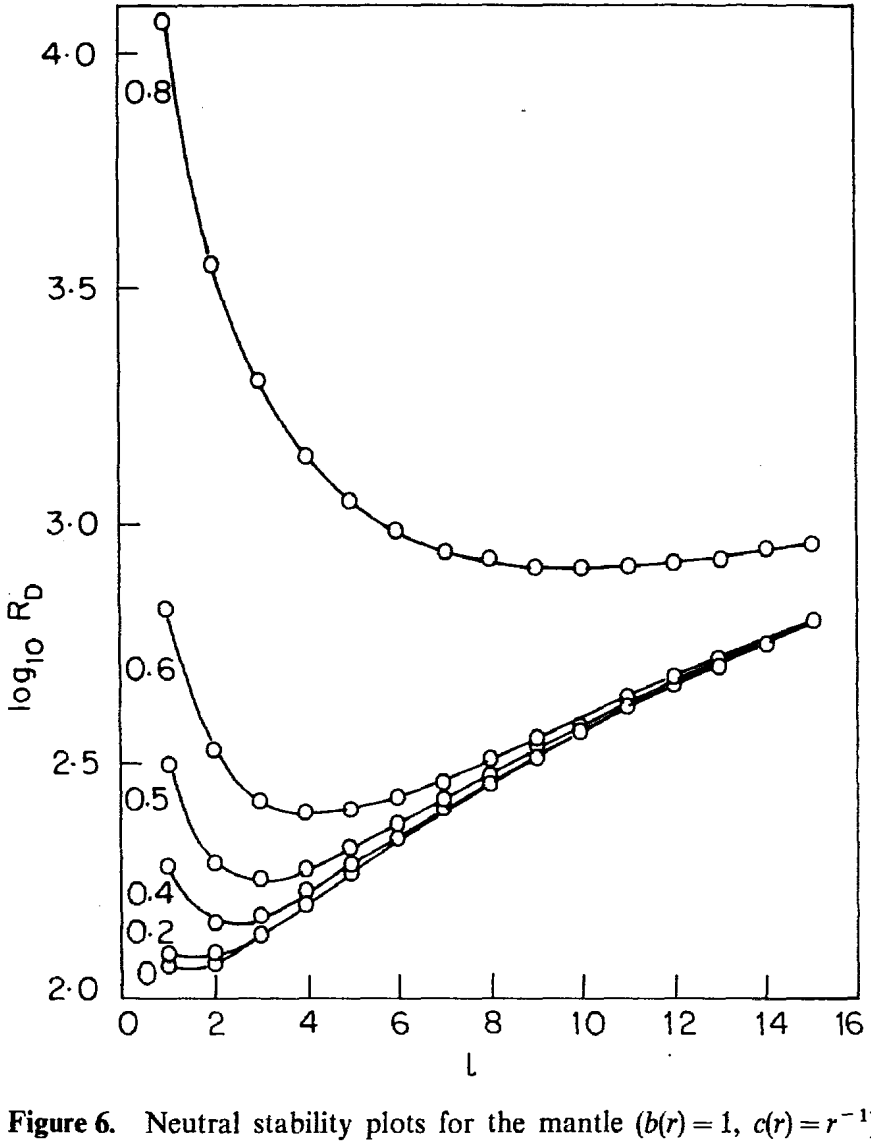


Figure 6. Neutral stability plots for the mantle ($b(r) = 1$, $c(r) = r^{-1}$) with impermeable boundary at $r = 1$ and permeable boundary at $r = \eta$.

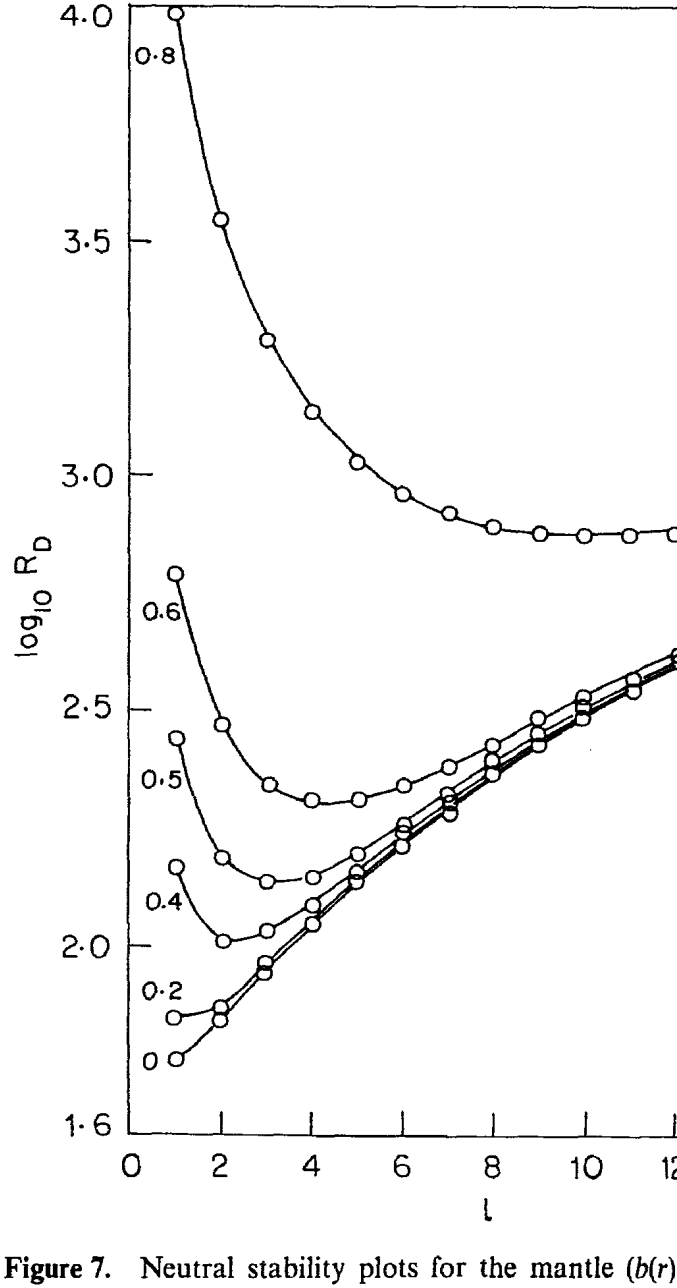


Figure 7. Neutral stability plots for the mantle ($b(r)$ with permeable boundary at $r = 1$ and impermeable bo

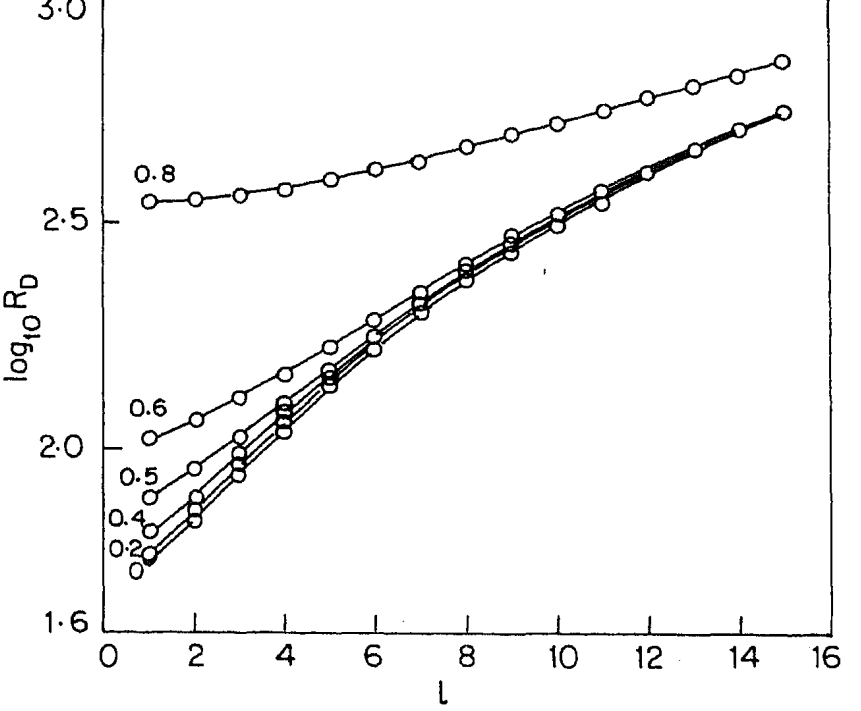


Figure 8. Neutral stability plots for the mantle ($b(r)=1$, $c(r)=r^{-1}$) with permeable boundaries both at $r=1$ and $r=\eta$.

Table 9. Characteristic number R_D for various values of l at $\eta=0$ with impermeable and permeable bounding surface at $r=1$ when gravity varies as r.

l	Impermeable ($\times 10^2$)	Permeable ($\times 10^2$)
1	2·03832	0·98090
2	1·83900	1·06266
3	1·98707	1·27857
4	2·24147	1·54919
5	2·55390	1·85801
6	2·90824	2·19920
7	3·29732	2·56998
8	3·71753	2·96887
9	4·16672	3·39487
10	4·64348	3·84727
11	5·14693	4·32550
12	5·67646	4·85730
13	6·23149	5·38786
14	6·81168	5·94338
15	7·41669	6·52328

Values of R_D for $l=1$ to 11 have been obtained

Table 10. Characteristic number R_D for various values of l at $\eta = 0$ with impermeable and permeable bounding surfaces at $r = 1$ when gravity is constant through the mantle.

l	Impermeable ($\times 10^2$)	Permeable ($\times 10^2$)
1	1·20261	0·57873
2	1·19167	0·68860
3	1·36313	0·87710
4	1·60265	1·10767
5	1·88478	1·37121
6	2·19863	1·66259
7	2·54553	1·98402
8	2·91826	2·33056
9	3·31671	2·70232
10	3·74265	3·10090
11	4·18960	3·52096
12	4·66605	3·99270
13	5·16591	4·46654
14	5·68775	4·96272
15	6·23743	5·48608

Further, the values of characteristic number R_D for $\eta = 0$ with either impermeable or permeable boundary for the first fifteen modes, when the gravity varies as r, obtained by Pradhan and Patra [12] and when gravity is constant through the mantle have been presented in tables 9 and 10 respectively and plotted in figures 1–4 and 5–8.

4.3. *Model of thermal convection in the mantle with a massive core*

We now consider a model of the mantle overlying a massive core so that the contribution of the mantle to the gravity can be totally ignored and there is uniform distribution of heat sources in the system. In the geophysical context, to a fair first approximation, the mantle can be considered as a spherical shell of fluid-saturated porous matter overlying a massive solid core, so that contribution of the mantle to the gravity can be considered negligible compared to that of the core. For this model, the radial gravitational field for a core of mass M_i is given by

$$rg(r) = \mathcal{G}\frac{M_i}{r^2} = (\text{constant})r^{-2}. \tag{68}$$

Equation (68) with the aid of relation

$$\alpha g(r) = \gamma(r) = \gamma_1 c(r) \tag{69}$$

gives

$$c(r) = r^{-3}, \tag{70}$$

where we have assumed that $\gamma_1 = \alpha\mathcal{G}M_i$. Since there is uniform distribution of heat sources, for the model under discussion, we have

The matrix element given in (49) with the aid of relation (71) reduces to

$$P_{k,j}^{1,c(r)} = \alpha_k^2 \int_\eta^1 \left\{ rC_{l+1/2,l+1/2}(\alpha_k r) \frac{1}{c(r)} C_{l+1/2,l+1/2}(\alpha_j r) \right\} \mathrm{d}r$$

$$= \alpha_k^2 \int_\eta^1 r^4 \{ C_{l+1/2,l+1/2}(\alpha_k r) C_{l+1/2,l+1/2}(\alpha_j r) \} \, \mathrm{d}r \tag{72}$$

Denoting

$$P_{k,j}^{1,c(r)} = \alpha_k^2 (k|c(r)^{-1}|j) = \alpha_k^2 (k|r^3|j) \tag{73}$$

the secular determinant (51) is obtained as

$$\left\| \alpha_k^2 (k|r^3|j) - l(l+1) R_D \left[\frac{N_{l+1/2,k}}{\alpha_k^2} \delta_{kj} - Q_{kj} \right] \right\| = 0 \tag{74}$$

and in the first approximation, we have

$$l(l+1) R_D = \frac{(1|r^3|1)\alpha_1^4}{N_{l+1/2,1} - Q_{11}\alpha_1^2}. \tag{75}$$

The values of the characteristic number R_D have been calculated from relation (75) for various values of l, for $\eta = 0{\cdot}5$ and for four sets of boundary conditions considered. They are tabulated in table 11 and the corresponding neutral stability plots have been illustrated in figure 9.

Table 11. Characteristic number R_D for $\eta = 0{\cdot}5$ and for different laws of variation of gravity and temperature distributions (either $b(r) = 1$, $c(r) = r^{-3}$ or $b(r) = r^{-3}$, $c(r) = 1$).

l	Perm. at $r = 1$ $r = \eta$ ($\times 10^2$)	Perm. at $r = 1$, Imp. at $r = \eta$ ($\times 10^2$)	Imp. at $r = 1$, Perm. at $r = \eta$ ($\times 10^2$)	Imp. at $r = 1$, $r = \eta$ ($\times 10^2$)
1	4·6144589*†	1·6201914	1·8494667	4·1292005
2	5·2656253 †	9·1897206†	1·1583410	1·9061807
3	6·2416493 †	8·2160987†	1·0744883*	1·4338549
4	7·4165087 †	8.6139405†	1·1369534	1·3370735*
5	9·1089848 †	9·8361469	1·2688958	1·3884052
6	1·0986330	1·1452453	1·4487992	1·5219367
7	1·3162442	1·3461006	1·6693605	1·7143502
8	1·5605619	1·5793891	1·9229668	1·9503695
9	1·8331306	1·8447571	2·2096342	2·2260667
10	2·1308233	2·1378312	2·5333043	2·5339502
11	2·4552142	2·4593125	2·8679979	2·8735510
12	2·8043270	2·8066991	3·2378665	3·2410027
13	3·1778860	3·1792209	3·6331264	3·6348630
14	3·5751278	3·5758654	4·0525166	4·0534560
15	3·9925100	3·9929037	4·4923044	4·4928070

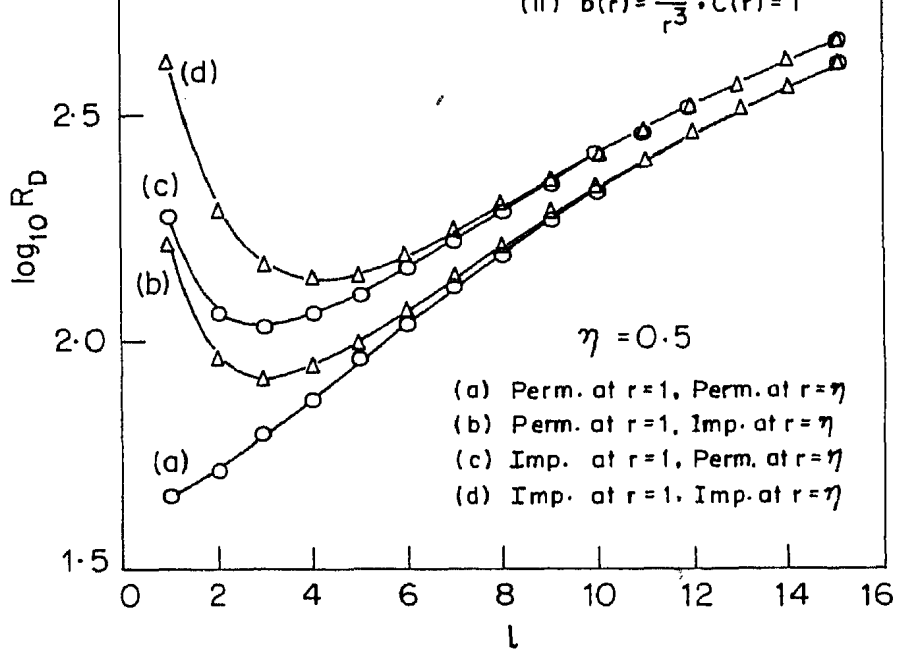


Figure 9. Darcy–Rayleigh number R_D for the onset of convection in the mantle as a spherical harmonic disturbance of order l for the shell thickness $\eta = 0{\cdot}5$ for various types of boundaries when $b(r) = 1$, $c(r) = r^{-3}$ or $b(r) = r^{-3}$, $c(r) = 1$.

4.4. *Model of the mantle when the gravity distribution is linear and all the heat sources are confined to the core*

For such a geophysical model where all the heat sources are confined to the core and none exists in the mantle and the variation of gravity is same as in a homogeneous sphere, from the expressions for $\beta(r)$, constancy of β_1 and linear gravity distribution, we obtain

$$b(r) = r^{-3} \quad \text{and} \quad c(r) = 1. \tag{76}$$

By (52), the matrix element given in (49) can be written as

$$P_{k,j}^{b(r),1} = P_{j,k}^{l,b(r)} = \alpha_j^2 (j|r^3|k) \tag{77}$$

and as such the secular determinant (51) now becomes

$$\left\| \alpha_j^2 (j|r^3|k) - l(l+1) R_D \left[\frac{N_{l+1/2,k}}{\alpha_k^2} \delta_{kj} - Q_{kj} \right] \right\| = 0. \tag{78}$$

Since Q_{kj} is symmetric in k and j, the secular determinant (78) by transposing rows and columns is the same as the determinant (74) with c replaced by b. Hence

$$R_D\{\text{for } c(r) = f(r) \text{ and } b(r) = 1\} = R_D\{\text{for } c(r) = 1 \text{ and } b(r) = f(r)\}. \tag{79}$$

The values of the characteristic number R_D for this model is therefore the same as for the model 4.3 presented in table 11 and illustrated in figure 9.

heat sources

As shown in (59) a constant value of gravity is strictly inconsistent with the mode of the mantle of constant density overlying a core. A variation of gravity compatible with (59) is now given by

$$rg(r) = \text{constant}\left(\frac{6}{7}r + \frac{1}{7r^2}\right), \quad (\text{when } \tfrac{1}{2} \leqslant r \leqslant 1). \tag{80}$$

The distribution of heat sources being uniform, we also have

$$b(r) = 1. \tag{81}$$

From (80) and (81) we obtain for the model under consideration

$$c(r) = \left(\frac{6 + r^{-3}}{7}\right), \quad \text{and} \quad b(r) = 1. \tag{82}$$

By (82), the value of the matrix element (49) is given by

$$P_{k,j}^{1,c(r)} = \alpha_k^2 \int_\eta^1 \left\{ rC_{l+1/2,l+1/2}(\alpha_k r) \frac{1}{c(r)} C_{l+1/2,l+1/2}(\alpha_j r) \right\} \mathrm{d}r$$

$$= \alpha_k^2 \int_\eta^1 \left\{ rC_{l+1/2,l+1/2}(\alpha_k r) \left(\frac{6 + r^{-3}}{7}\right)^{-1} C_{l+1/2,l+1/2}(\alpha_j r) \right\} \mathrm{d}r. \tag{83}$$

Table 12. Characteristic number R_D for $\eta = 0{\cdot}5$ and for laws of variation of gravity and temperature distribution: $c(r) = (6 + r^{-3})/7$ and $b(r) = 1$.

l	Perm. at $r = 1$, $r = \eta$ ($\times 10^2$)	Perm. at $r = 1$, Imp. at $r = \eta$ ($\times 10^2$)	Imp. at $r = 1$, Perm. at $r = \eta$ ($\times 10^2$)	Imp. at $r = 1$, $r = \eta$ ($\times 10^2$)
1	8·6091764*†	3·0227843	3·4505423	7·7038319
2	9·7589595†	1·7031616	2·1467921	3·5327884
3	1·1440644	1·5059715*	1·9694857*	2·6281874*
4	1·3407913	1·5572687	2·0554379	2·4172244*
5	1·6212123	1·7506323	2·2583742	2·4710764
6	1·9209903	2·0024933	2·5332658	2·6611487
7	2·2560936	2·3072687	2·8613486	2·9384627
8	2·6240633	2·6557210	3·2334421	3·2795194
9	3·0226790	3·0418500	3·6435019	3·6705976
10	3·4474762	3·4588144	4·0986535	4·0996986
11	3·9034517	3·9099674	4·5597209	4·5685560
12	4·3816692	4·3853756	5·0590605	5·0639607
13	4·8864267	4·8884793	5·5864199	5·5890910
14	5·4170035	5·4181212	6·1403391	6·1417632
15	5·9706851	5·9712739	6·7181134	6·7188652

* Denotes the minimum value of R_D. Perm. = Permeable; Imp. = Impermeable.

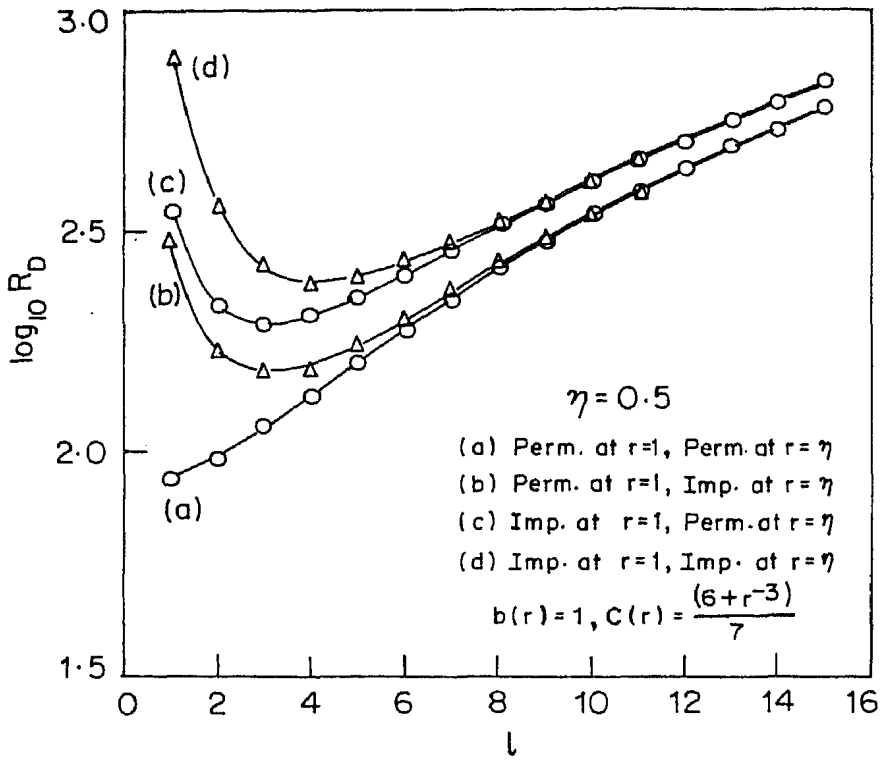


Figure 10. Darcy–Rayleigh number R_D for the onset of convection in the mantle as a spherical harmonic disturbance of order l for the shell thickness $\eta = 0{\cdot}5$ for various types of boundaries when $b(r) = 1$, $c(r) = (6 + r^{-3})/7$.

Denoting

$$P_{k,j}^{1,c(r)} = \alpha_k^2 (k|c(r)^{-1}|j) \tag{84}$$

the secular determinant (51) reduces to the form

$$\left\| \alpha_k^2 (k|c(r)^{-1}|j) - l(l+1) R_D \left[\frac{N_{l+1/2,j}}{\alpha_j^2} \delta_{kj} - Q_{kj} \right] \right\| = 0, \tag{85}$$

where

$$(k|c(r)^{-1}|j) = \int_{1/2}^{1} \left\{ \left(\frac{7r^4}{6r^3+1} \right) C_{l+1/2,l+1/2}(\alpha_k r) C_{l+1/2,l+1/2}(\alpha_j r) \right\} \mathrm{d}r. \tag{86}$$

In the first approximation, we obtain

$$l(l+1) R_D = \frac{(1|c(r)^{-1}|1)\alpha_1^4}{N_{l+1/2,1} - Q_{11}\alpha_1^2} \tag{87}$$

which gives the values of R_D for $\eta = 0{\cdot}5$, for various values of l and for the set of four boundary conditions. These values are presented in table 12 and further illustrated in figure 10.

mantle gradually recedes to a spherical harmonic of higher order as the mantle thickness decreases. This is true in all the three cases of the shell having both impermeable, impermeable-permeable and permeable-impermeable bounding surfaces. But, in contrast, in the permeable-permeable case the onset of thermal convection occurs at spherical harmonic of order 1, irrespective of the thickness of the mantle. The first three cases are in harmony with the corresponding (ridig and/or free bounding) cases of the continuous fluid, where l increases with the decreasing mantle thickness, as established by Chandrasekhar [5]. But there is a difference in case both the bounding surfaces are permeable, as the lowest harmonic occurs at $l = 1$, in contrast to that of homogeneous fluid with free-free boundaries.

The result for the third and fourth models [cf. §4.3 and §4.4] is the same because of the identity of the underlying characteristic value problem. From table 11 and figure 9 it is seen that for a particular value of the thickness of the shell, $\eta = 0{\cdot}5$, the order of the spherical harmonic shifts from 1 when both the surfaces are permeable, to 3 when one surface is permeable and the other impermeable, and finally to 4 when both surfaces are impermeable. For the fifth model of the mantle with constant density and with uniform distribution of heat sources for radius ratio $\eta = 0{\cdot}5$, the dependence of critical Darcy–Rayleigh number for the onset of convection on the order of the spherical harmonic is similar to models III and IV.

The values of R_D for all the five models discussed here are comparatively lower than those for the continuous fluid shell and merit comparison with available data.

It has been argued that the mantle with fluid magma and semisolid rocks behave similarly as a porous medium saturated with a fluid. The main difference between the continuous fluid model discussed in Chandrasekhar [3] and the porous fluid model discussed in this paper seem to be the sensitivity to kinematic boundary conditions that the porous model exhibits. This brings into sharp focus a number of qualitative differences between the two models. For example, consider the manner of the onset of thermal instability in a porous mantle. In the case of a porous mantle occupying half the radius of the sphere, we have seen that if both the boundaries are permeable, the current system must belong principally to the mode $l = 1$, instead to the modes $l = 3$, 4 and 5 as in a homogeneous fluid shell or to the modes $l = 3$, 4 and 6 as in a porous fluid shell with impermeable boundaries or to the modes $l = 2, 3$ and 4 as in a porous fluid shell with one of the boundaries permeable and the other impermeable.

It is further observed from the figures that in all the five models discussed in this paper the neutral stability plots for different mantle thickness do not cross but lie symmetrically one above the other. The critical values of the Rayleigh number for the models discussed here are comparatively lower than those for the continuous fluid shell and merit comparison with available data.

Acknowledgement

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Vertical vibration of a circular stamp on an elastic-porous half-space with saturation fluid

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Abstract. A specific problem concerning the vertical vibrating response of a composite layer (elastic-porous and saturation fluid) under rigid circular stamp is studied. The motion of the medium is governed by *Biot's* dynamic potential equations and the problem is solved by the method of Fourier-Bessel transformation. Formulae are obtained for the vertical displacement of the rigid phase. Graphs are presented for the amplitude of vibration.

Keywords. Composite materials; elasticity; porous media; vibration; saturation fluid.

1. Introduction

The method of *Fourier-Bessel* transform with the asymptotic method are used to obtain a solution to the problem of circular stamp vibrating on a half space of a composite layer, with a circular contact surface. A large number of papers have already been published dealing with the mechanics of composite layers. Using the Biot [1] equations, the formulation and methods of solution of the fundamental boundary value problems can be found for such media in the monographs [7, 8]. For a rigorous interaction problem dealing with the method of solving contact stamp problems for composite layer, attention should be drawn to [2, 3, 4, 5, 6].

2. Formulation of the problem

Two-dimensional problem of the vertical vibration of the circular rigid stamp are covered by the potential equations of Biot [1] given by

$$\begin{aligned}
&S_{11}\ddot{\phi}_{10} + S_{12}\ddot{\phi}_{20} + b(\dot{\phi}_{10} - \dot{\phi}_{20}) = P\Delta\phi_{10} + Q\Delta\phi_{20}\\
&S_{12}\ddot{\phi}_{10} + S_{22}\ddot{\phi}_{20} - b(\dot{\phi}_{10} - \dot{\phi}_{20}) = Q\Delta\phi_{10} + R\Delta\phi_{20}\\
&S_{11}\ddot{\Psi}_{10} + S_{12}\ddot{\Psi}_{20} + b(\dot{\Psi}_{10} - \dot{\Psi}_{20}) = N\Delta_1\Psi_{10}\\
&S_{12}\ddot{\Psi}_{10} + S_{22}\ddot{\Psi}_{20} - b(\dot{\Psi}_{10} - \dot{\Psi}_{20}) = 0
\end{aligned} \tag{1}$$

where dot means a differentiation with respect to time.

$$\Delta = \partial_{,rr} + r^{-1}\partial_{,r} + \partial_{,r} + \partial_{,zz} \quad \text{and} \quad \Delta_1 = \partial_{,rr} + r^{-1}\partial_{,r} - r^{-2} + \partial_{,zz}.$$

The coordinates (r, Z) are in the cylindrical system where Z is directed down in the layer. $\partial_{,r}$ means differentiation with respect to r, and S_{11}, S_{22} and S_{12} are the dynamic coefficients satisfying the following relations

$$S_{11} + S_{12} = mS_f = S_1; \quad S_{22} + S_{12} = (1-m)S_s = S_2$$

S_f and S_s are the densities of the fluid and solid respectively; $b = (n/k)m^2$ (n is the viscosity of the fluid, k the permeability coefficient and the m porosity).

A, Q, N and R are the elastic constants characterizing the mechanical properties of the medium (fluid + solid) [1]. ϕ_{10}, ϕ_{20} are the potential functions of the longitudinal wave and Ψ_{10} and Ψ_{20} are the potentials of the transverse wave.

Let $\{u_r, u_z\}$ be the displacement component of the solid and $\{v_r, v_z\}$ the displacement component of the fluid; then we have

$$u_r = \phi_{10,r} - \Psi_{10,z}; \quad u_z = \phi_{10,z} + \Psi_{10,r} + r^{-1}\Psi_{10}$$

$$v_r = \phi_{10,r} - \Psi_{20,z}; \quad v_z = \phi_{20,z} + \Psi_{20,z} + r^{-1}\Psi_{20} \tag{2}$$

For the harmonic oscillation with complex frequency w, (1) can be re-written as [7]

$$(\Delta + K_1^2)\phi_1 = 0; \quad (\Delta + K_2^2)\phi_2 = 0, \quad (\Delta_1 + K_3^2)\Psi_1 = 0, \tag{3}$$

where

$$\phi_{10} = (\phi_1 + \phi_2)\exp(iwt); \quad \phi_{20} = (n_1\phi_1 + n_2\phi_2)\exp(iwt)$$

$$\Psi_{10} = \Psi_1 \exp(iwt); \quad \Psi_2 = S_0\Psi_1$$

$$C_{1,2}^2 = \frac{G + iB_0H \pm \sqrt{(G + iB_0H)^2 - 4B_1(g + iB_0S)}}{(g + iB_0S)}, \quad K_j = \left(\frac{w}{C_j}\right)^2, \quad j = 1, 2,$$

$$C_3^2 = \frac{S_{11} + S_0S_{12} - iB_0(1 - S_0)}{N}, \quad S_0 = -\frac{S_{12} + iB_0}{S_{22} - iB_0}; \quad B_0 = \left(\frac{b}{w}\right)$$

$$G = PS_{22} + RS_{11} - 2QS_{12}$$

$$H = P, \quad 2Q + R, \quad S = S_{11} + S_{22} + 2S_{12}, \quad B_1 = PR - Q^2$$

$$n_j = \frac{p - c_j^2(S_{11} + iB_0)}{(S_{12} - iB_0)c_j^2 - Q} \quad j = 1, 2.$$

Let the radius of the circular stamp a_0 which perform vertical harmonic oscillation with frequency w, and the contact between the surface of the half space is complete. For neglecting the shear stress at the surface, the boundary conditions are given by

$$\begin{aligned} &\delta_{rz} = \delta_{zz} = \delta = 0 \quad a_0 < r < \infty, \quad z = 0 \\ &\delta_{rz} = \delta = 0 \qquad\quad 0 < r < a_0, \quad z = 0 \end{aligned} \tag{4}$$

where δ_{rz}, δ_{zz} are the components of the stress tensor and $-\delta = mp_f$: P_f is the pressure of the fluid.

$$K_1^2 Q_1 \phi_1 + K_2^2 Q_2 \phi_2 = 0 \quad z = 0; \quad 0 < r < \infty \tag{5}$$

$$2\phi_{1,rz} + 2\phi_{2,rz} - K_3^2 \Psi_1 - 2\Psi_{1,zz} = 0 \quad z = 0; \quad 0 < r < \infty$$

$$\phi_{1,z} + \Psi_{1,r} + r^{-1}\Psi_1 = u_0 \quad z = 0 \quad 0 < r < a_0$$

$$A_j = A + n_j Q; \quad Q_j = Q + n_j R \quad j = 1, 2.$$

3. Solution of the problem

The solution of the problems (1)–(5) is contingent on the determination of δ_{zz} under the stamp. In the article [3] the assumption was

$$\begin{aligned} \delta_{zz} &= \frac{P_0}{\pi a_0^2} f(r) \quad 0 < r < a_0 \\ \delta_{zz} &= 0 \qquad\qquad a_0 < r < \infty \end{aligned} \tag{6}$$

or

$$-K_1^2 A_1 \phi_1 - K_2^2 A_2 \phi_2 + \mu(\phi_{1,zz} + \phi_{2,zz} + \Psi_{1,rz} + r^{-1}\Psi_{1,z})$$

$$= \begin{cases} \dfrac{P_0}{\pi a_0^2} f(r) & a < r < a_0 \\ 0 & a_0 < r < \infty. \end{cases}$$

Now the solution is contingent on the determination of the arbitrary function $f(r)$ which in turn depends on the spatial boundary conditions (5) and the integration of it on a circular area is equal to one. The mixed boundary value problem comprising of partial differential equation (3) and boundary conditions (5) and (6) is conveniently handled by means of integral transform. As a result of cylindrical geometry the obvious choice is the *Fourier-Bessel* transform which denote

$$\phi_j = \int_0^\infty \lambda D_j(\lambda) \exp(-j^{\alpha z}) \quad J_0(\lambda r)\, \mathrm{d}\lambda \quad j = 1, 2$$

$$\Psi_j = \int_0^\infty \lambda D_3(\lambda) \exp(-\alpha 3^z) J_1(\lambda r)\, \mathrm{d}\lambda$$

where $\alpha_j = \sqrt{\lambda^2 - K_j^2}$, $j = 1, 2, 3$ and J_0 and J_1 are the Bessel functions of the first kind of order 0 and 1 respectively; D_j is the determinate from the boundary conditions (4) and (5):

$$D_1 = \frac{p_0}{\pi a_0^2 \mu} \frac{(2\lambda^2 - K_3^2)\bar{H}_0(\lambda a_0)}{M(\lambda)}$$

$$D_2 = -\frac{DP_0}{\pi a_0^2 \mu} \frac{(2\lambda^2 - K_3^2)\bar{H}_0(\lambda a_0)}{M(\lambda)}$$

$$D_3 = \frac{2p_0 \lambda}{\pi a_0^2 \mu} \frac{(\alpha_1 - D\alpha_2)\bar{H}_0(\lambda a_0)}{M(\lambda)}$$

$$m_j = \frac{(\lambda + D_j Q)c_3^2}{c_j^2 \mu}; \quad j = 1, 2$$

$$D = \frac{c_2^2 Q_1}{c_1^2 Q_2}, \quad \bar{H}_0(\lambda a_0) = \int_0^{a_0} r f(r) J_0(\lambda r)\, dr.$$

The vertical displacement component of the rigid phase at $z = 0$ is given by

$$u_z = \frac{p_0 \exp(iwt)}{\pi a_0 \mu} \left\{ \int_0^{\infty} \frac{K_3^2(\alpha_1 - D\alpha_2)\bar{H}_0(\lambda a_0) J_0(\lambda r)\, d\lambda}{M(\lambda_0)} \right.$$

$$\left. - \frac{\pi i k_3^2(\alpha_1^0 - D\alpha_2^0)\bar{H}_0(\lambda_0 a_0) J_0(\lambda_0 r)}{M_1(\lambda_0)} \right\} \tag{7}$$

where $\alpha_j^0 = \sqrt{\lambda_0^2 - K_j^2}$; λ_0 are the roots of the equation

$$M(\lambda) = 0; \quad \text{and} \quad M_1(\lambda_0) = M, \lambda/\lambda = \lambda_0.$$

From (7) it is clear that the displacement component depends on the function $f(r)$ and from article [8] we have a complete contact between the stamp and the half space. The displacement of them are equal at the point where $z = 0$ and $r = 0$. Then (7) becomes

$$u_z = \frac{p_0 \exp(iwt)}{\pi a_0 \mu} (f_1 + i f_2) \tag{8}$$

where f_1 and f_2 are the real functions. If the mass of the stamp is M_0 then the amplitude of vibration of the rigid stamp on elastic porous half space under the periodic concentrate force p_0 is given by [4]

$$A = \frac{p_0}{\mu a_0} \left[\frac{f_1^2 + f_2^2}{-(1 + \bar{M}_0 \bar{w} f_1)^2 + (\bar{M}_0 \bar{w}^2 f_2)^2} \right]^{1/2},$$

$$\bar{M}_0 = \frac{M_0}{S a_0^3}; \quad \bar{w} = w a_0 \left(\frac{S_{11} S_{22} - S_{12}^2}{S_{22} \mu} \right)^{1/2}.$$

Functions f_1 and f_2 can be calculated by using the approximation methods or the computational techniques. For $b = 0$ the parameters D_j, S_0, n_j and D are real. Introducing the dimensional values

$$\bar{P} = a_0 P, \; l_j = \frac{c_3}{c_j} (j = 1, 2)$$

and for the case when $l_1 < l_2 < l$ we have

$$f_1 = \int_{l_1}^{l_2} \frac{(\alpha_1 q_1 + \bar{\alpha}_2 D q_2)\bar{H}_0(\bar{\lambda})\, d\lambda}{q_1^2 + q_2^2} + \int_{l_2}^{l} \frac{(\alpha_1 - D\alpha_2) E_1 \bar{H}_0(\bar{\lambda})\, d\bar{\lambda}}{E_1^2 + E_2^2}$$

$$\int^{\infty} (\alpha_1 - D\alpha_2)\bar{H}_0(\bar{\lambda})\, d\bar{\lambda}$$

$$+\int_{l_2}^{l}\frac{(\alpha_1-D\alpha_2)E_2\bar{H}_0(\bar{l})\,\mathrm{d}\lambda}{E_1^2+E_2^2}-\frac{\bar{w}^2(\alpha_1^0-D\alpha_2^0)\bar{H}_0(\bar{\lambda}_0)}{F_{22}(\bar{\lambda}_0)}$$

where

$$\alpha_j=\sqrt{\bar{\lambda}^2-l_j^2}\,(j=1,2)\quad \alpha_3=\sqrt{\bar{\lambda}^2-1}$$

$$\bar{\alpha}_j=\sqrt{l_j^2-\bar{\lambda}^2}\,(j=1,2)\quad \bar{\alpha}_3=\sqrt{1-\bar{\lambda}^2}$$

$$q_1=(2\bar{\lambda}^2-1)^2(1-D)-4\bar{\alpha}_3\bar{\alpha}_2\bar{\lambda}^2D$$

$$q_2=4\bar{\alpha}_3\alpha_1\bar{\lambda}^2;\quad E_1=(2\bar{\lambda}^2-1)^2(1-D)$$

$$E_2=4\bar{\alpha}_3\bar{\lambda}^2(\alpha_1-D\alpha_2)$$

$$F_1=E_1-4\alpha_3\bar{\lambda}^2(\alpha_1-D\alpha_2)$$

$$F_2=E_1+4\alpha_3\bar{\lambda}^2(\alpha_1-D\alpha_2)$$

$$D=\frac{D_1^0}{D_2^0};\quad D_j^0=-\frac{P-S_{11}C_j^2}{Q-S_{12}C_j^2}\quad (j=1,2)$$

$\bar{\lambda}_0$ are the roots of the equation $F_2(\bar{\lambda})=0$

$$F_{22}(\bar{\lambda}_0)=F_2,\quad \lambda/\lambda=\bar{\lambda}_0;\quad \alpha_j^0=\sqrt{B_j^2-\bar{\lambda}_0^2}\quad (j=1,2)$$

In the case when $e=(ws/b)^{1/2}\ll 1$. From (7) we have for the first order only for

$$u_z=\frac{p_0\exp(iwt)}{\mu\pi a_0}\int_0^{\infty}\frac{K_{33}^2\alpha_{11}\bar{H}_0(\lambda a_0)J_0(\lambda_{01}r)\,\mathrm{d}\lambda}{M_{11}(\lambda)}$$

$$-\frac{\pi iK_{33}^2\alpha_{10}\bar{H}_0(\lambda_0 a_0)J_0(\lambda_{01}a)}{M'_{11}(\lambda_0)}$$

$$+\frac{\sqrt{2}}{2}eK_{33}^3i(l+i)\int_0^{\infty}\frac{(2\lambda^2-K_{33}^2)^2a\bar{H}_0(\lambda_0 a)J_0(\lambda_{01}a)\,\mathrm{d}\lambda}{M_{11}^2(\lambda)},$$

where

$$K_{j3}=\frac{w}{C_{j3}};\quad \alpha_{j1}=\sqrt{\lambda^2-K_{11}^2}\quad (j=1,3)$$

$$C_{33}=\sqrt{\frac{M}{S}};\quad C_{11}=\sqrt{\frac{H}{S}}$$

$$M_{11}(\lambda)=(2\lambda^2-K_{33}^2)-4\lambda^2\alpha_{11}\alpha_{33}$$

λ_{01} are the roots of the equation $M_{11}(\lambda)=0$

$$M'_{11}(\lambda_0)M,\lambda/\lambda=\lambda_0\quad \text{and}\quad \alpha_{10}=\sqrt{\lambda_{10}^2-k_{11}^2}.$$

Equations (8) and (9) are identical at the points $z = 0$, $r = 0$ and the displacement component is the sum of the two terms, (i) when $e = 0$ express the motion of the rigid and fluid phase and (ii) express the interchange motion of the two components of the fluid and rigid phases, when the fluid has a high viscosity, or when the medium has a low permeability.

4. Numerical calculation

1. We shall express the dynamic elastic constants A, N, Q and R through the mechanical properties of the medium as [1],

$$A = \lambda_e + 2\mu + \frac{hR_0a^2}{m}; \quad N = \mu; \quad Q = haR_0$$

$$h = (mk_0/K_c) + aR_0; \quad a = 1 - m - (K_c/K_0),$$

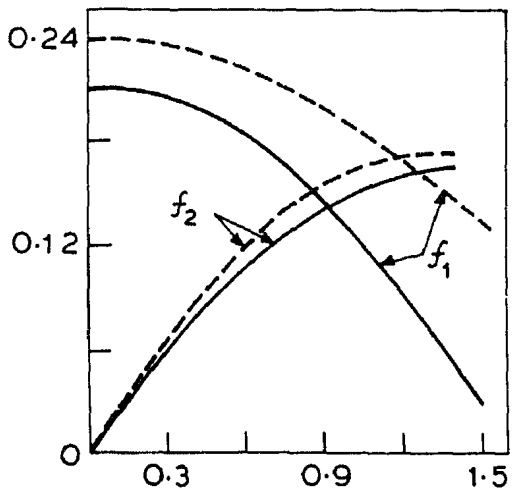
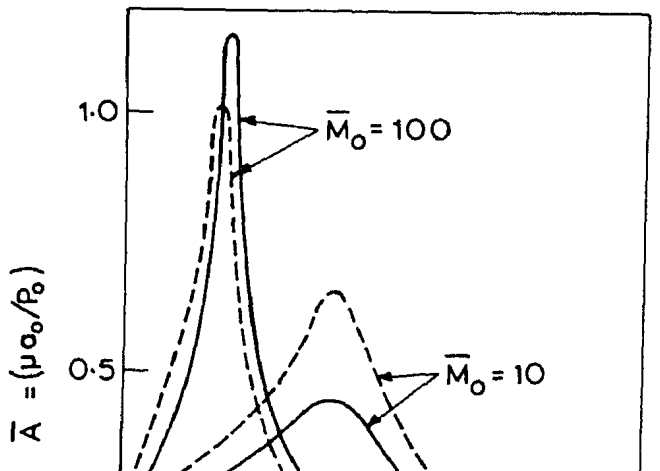


Figure 1.



where λ_e and μ are the Lame constants, K_c the compression volume model of porous skeleton, K_0 the compressibility proper model of the rigid skeleton and R_0 the compressibility model of the fluid.

2. When $b = 0$; $m = 0,2$; $(R_0/K_0) = 0,2$; $(K_0/K_c) = 0,1$; $S_{12} = 0$ and $S_f/S_s = 0,4$. We have the change of the functions f_1 and f_2 with $\bar{w} = wa_0/c_3$ for fixed values of $\bar{M}_0$ as shown in figure 1. The shape of the amplitude $\bar{A} = (\mu a/p_0)A$ when Poisson's coefficient of the medium has value 0·3 presented in the figure 2. The dotted lines represent the linear elastic medium.

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Rayleigh wave scattering by a plane strip in a deep ocean

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Abstract. A theoretical formulation to study the problem of scattering of Rayleigh waves due to the presence of a rigid plane strip in a deep ocean is presented. A rigid plane strip ($0 \leqslant z \leqslant H$, $0 \leqslant x \leqslant l$) is fixed in the surface of the ocean occupying $z \geqslant 0$. Fourier transformation and Wiener–Hopf technique are used to arrive at the solution. The scattered Rayleigh waves behave as cylindrical waves emerging out of the corner of the strip and its image in the free surface of the ocean. The scattered waves are obtained in terms of Bessel functions whose behaviour near and far from the strip is well-known. The numerical calculations for the scattered waves show that their amplitude increases rapidly for a small increase in the value of the wave number. Scattering of Rayleigh waves due to a thin plane vertical barrier and a thin barrier in the free surface of the ocean has been considered as the special cases.

Keywords. Rayleigh wave scattering; rigid plane strip; Fourier transformation; Wiener–Hopf technique.

1. Introduction

The study of Rayleigh wave scattering due to surface discontinuities has acquired importance as they are responsible for the loss of human lives and damage of buildings on the earth surface during an earthquake. As soon as the seismic waves appear on the earth surface, they let loose their energies around the inhomogeneities and irregularities in the earth surface and display their catastrophic role.

The vertical barrier effect, fixed in an infinitely deep sea, on normally incident surface waves was first considered by Ursell [9] for a two-dimensional case. Deshwal [1] extended this study to consider the problem of diffraction of compressional waves due to a thin plane barrier in the deep seawater surface using the technique of Wiener and Hopf [6]. Scattering of Rayleigh waves at the corner of an elastic quarter space has been considered by Momoi [3] using the technique of Fourier transformation. He approximated the integral transformations to obtain the expressions for the energies of Rayleigh waves along the two free surfaces. He [4, 5] also studied the problem of scattering of Rayleigh waves by semi-circular and rectangular discontinuities in the surface of a solid half space. The solutions are not exact because of approximations used in the solution of these problems. The effect of topography and subsurface inhomogeneity on seismic Rayleigh waves has been investigated by Ohtsuki and Yamashara [8]. Several types of topography such as cliffs are considered and computations are made using a hybrid method combining a particle model with a finite element method.

We intend to discuss here the problem of diffraction of Rayleigh waves at the edge of a rigid plane strip. The strip is rigid such that no displacement takes place across it. It is a two-dimensional problem.

2. Statement of the problem

The liquid half space is formed by zx-plane with x-axis along the free surface and z-axis pointing vertically downwards along the depth of the liquid. The two-dimensional time-harmonic Rayleigh waves are incident at the strip $0 \leqslant x \leqslant l, 0 \leqslant z \leqslant H$ in the zx-plane. The wave equation in two dimensions is

$$\partial^2\bar{\phi}/\partial z^2 + \partial^2\bar{\phi}/\partial x^2 = [\partial^2\bar{\phi}/\partial t^2 + \varepsilon\partial\bar{\phi}/\partial t]/c^2, \tag{1}$$

where c is the velocity of propagation of Rayleigh wave and $\varepsilon > 0$ the damping constant. The potential of the time-harmonic wave is taken as

$$\bar{\phi}(x, z, t) = \phi(x, z)\exp(-i\omega t). \tag{2}$$

Now the wave equation (1) reduces to

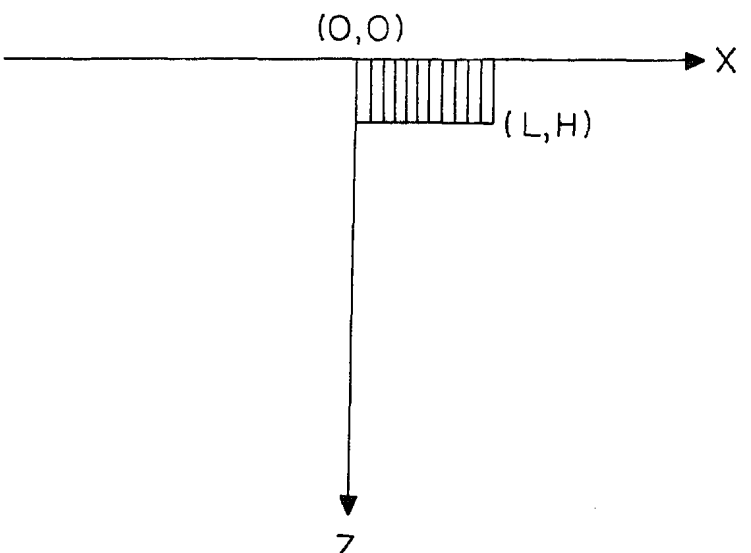
$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial z^2 + k^2\phi = 0, \tag{3}$$

where

$$k = (\omega^2 + i\varepsilon\omega)^{1/2}/c = k_1 + ik_2. \tag{4}$$

The imaginary part of k i.e. k_2 is positive and small. The incident wave satisfying the free surface condition is

$$\phi_i = A_0 \sinh \gamma_0 z \exp(i\alpha_0 x), \quad \gamma_0 = \pm(\alpha_0^2 - k^2)^{1/2}, \tag{5}$$



where α_0 is the root of Rayleigh equation. Let

$$\phi_t = \phi_i + \phi \tag{6}$$

represent the total potential at any point (x, z) inside the liquid half space. We assume that, for given z

$$|\phi| \sim M \exp(-k_2|x|), \quad \text{as } x \to \infty, \quad M > 0. \tag{7}$$

The Fourier transformations are defined as

$$\bar{\phi}(\alpha, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x, z) \exp(i\alpha x)\, dx = \int_{-\infty}^{0} + \int_{0}^{l} + \int_{l}^{\infty}$$

$$= \bar{\phi}_-(\alpha, z) + \bar{\phi}_l(\alpha, z) + \bar{\phi}_{+l}(\alpha, z). \tag{8}$$

If $\alpha = \sigma + iT$, then from (8), we obtain

$$|\bar{\phi}_+(\alpha, z)| \leqslant \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} |\phi| \exp(-Tx)\, dx. \tag{9}$$

On account of (7), $\bar{\phi}_+(\alpha, z)$ is bounded at $x = \infty$, only when $T > -k_2$. Thus $\bar{\phi}_+(\alpha, z)$ is regular in the region $T > -k_2$ of the complex α-plane. Therefore

$$|\bar{\phi}| \leqslant |\bar{\phi}_+| + |\bar{\phi}_-| \tag{10}$$

implies that $\bar{\phi}(\alpha, z)$ is regular in the strip $-k_2 < T < k_2$ of the α plane. Similarly its derivatives w.r.t. z will be regular in their respective regions of analyticity.

3. Boundary conditions

The conditions to be satisfied on the boundaries are

(i) $$\phi_t = 0, \quad z = 0, \quad -\infty < x \leqslant 0, \quad l \leqslant x < \infty, \tag{11}$$

(ii) $$U = \partial\phi_t/\partial x = 0, \quad 0 \leqslant z \leqslant H, \quad x = 0, l, \tag{12}$$

(iii) $$W = \partial\phi_t/\partial z = 0, \quad z = H, \quad 0 \leqslant x \leqslant l, \tag{13}$$

where (U, W) are displacement components at any point (x, z) of the medium. The last two conditions imply that at the points $(0, H)$ and (l, H), ϕ_t is merely a constant and is taken equal to be zero.

4. Solution of the problem

We apply Fourier transform to (3) for $x = -\infty$ to $x = \infty$ noting that ϕ and $\partial\phi/\partial x$ vanish at $x = \pm\infty$, we obtain

The sign before the radical in (14) is chosen so that the real part of $\gamma \geqslant 0$ for all α. Solving (14), we get

$$\bar{\phi}(\alpha, z) = A(\alpha)\exp(-\gamma z) + B(\alpha)\exp(\gamma z). \tag{15}$$

Since $\phi(x, z)$ and hence $\bar{\phi}(\alpha, z)$ is bounded as $z \to \infty$, we take $B(\alpha) = 0$. Thus (15) results in

$$\bar{\phi}(\alpha, z) = A(\alpha)\exp(-\gamma z), \quad z \geqslant H. \tag{16}$$

Eliminating $A(\alpha)$ between (16) and its derivative w.r.t. z, we obtain

$$\bar{\phi}(\alpha, z) = -\gamma^{-1}\bar{\phi}'(\alpha, z). \tag{17}$$

Putting $z = H$ in (17) and using the notations $\bar{\phi}(\alpha)$, $\bar{\phi}'(\alpha)$ for $\bar{\phi}(\alpha, H)$, $\bar{\phi}'(\alpha, H)$ respectively, we see

$$\bar{\phi}(\alpha) = -\gamma^{-1}\bar{\phi}'(\alpha) \quad \text{or} \quad \bar{\phi}_+(\alpha) + \bar{\phi}_-(\alpha) = -\gamma^{-1}[\bar{\phi}'_+(\alpha) + \bar{\phi}'_-(\alpha)]. \tag{18}$$

The functional equation (18) is to be solved by the Wiener–Hopf technique. We decompose (18) as

$$\bar{\phi}_+(\alpha) + \frac{1}{(\alpha - k)^{1/2}}\left(\frac{\bar{\phi}'_+(\alpha)}{(\alpha + k)^{1/2}} - \frac{\bar{\phi}'_+(k)}{(2k)^{1/2}}\right) + \frac{\bar{\phi}'_-(-k)}{[-2k(\alpha + k)]^{1/2}}$$

$$= -\bar{\phi}_-(\alpha) - \frac{1}{(\alpha + k)^{1/2}}\left(\frac{\bar{\phi}'_-(\alpha)}{(\alpha - k)^{1/2}} - \frac{\bar{\phi}'_-(-k)}{(-2k)^{1/2}}\right) - \frac{\bar{\phi}'_+(k)}{[2k(\alpha - k)]^{1/2}}. \tag{19}$$

Rewriting (19) as

$$\bar{\phi}_+(\alpha) + \gamma^{-1}\bar{\phi}'_+(\alpha) - G(\alpha) = -\bar{\phi}_-(\alpha) - \gamma^{-1}\bar{\phi}'_-(\alpha) - G(\alpha), \tag{20}$$

where

$$G(\alpha) = \frac{\bar{\phi}'_+(k)}{[2k(\alpha - k)]^{1/2}} - \frac{\bar{\phi}'_-(-k)}{[-2k(\alpha + k)]^{1/2}}. \tag{20a}$$

The left hand member of (20) is analytic in the region $T > -k_2$ and the right hand member is analytic in the region $T < k_2$ so that $-k_2 < T < k_2$ is the common region of analyticity. Each member is an analytic continuation of the other and equal to a function, say $F(\alpha)$. $F(\alpha) \to 0$ as $\alpha \to \infty$. By extended form of Liouville's theorem, each member in (20) is zero. Thus

$$\bar{\phi}_+(\alpha) = -\gamma^{-1}\bar{\phi}'_+(\alpha) + G(\alpha), \quad \bar{\phi}_-(\alpha) = -\gamma^{-1}\bar{\phi}'_-(\alpha) - G(\alpha). \tag{21}$$

Similarly,

$$\bar{\phi}_+(-\alpha) = -\gamma^{-1}\bar{\phi}'_+(-\alpha) + G(-\alpha), \quad \bar{\phi}'_-(-\alpha) = -\gamma^{-1}\bar{\phi}'_-(\alpha) - G(-\alpha). \tag{21a}$$

We multiply (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrate w.r.t. x between $x = -\infty$ and $x = 0$, to find

$$d^2[\bar{\phi}_-(\alpha, z)]/dz^2 - \gamma^2[\bar{\phi}_-(\alpha, z)] = (-\partial\phi/\partial x + i\alpha\phi)/(2\pi)^{1/2} \quad \text{at } x = 0.$$

Changing α to $-\alpha$ in (22) and adding the result to it, we obtain

$$d^2[\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z)]/dz^2 - \gamma^2[\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z)]$$
$$= -2(\partial\phi/\partial x)/(2\pi)^{1/2} \quad \text{at } x = 0. \tag{23}$$

The value of $\partial\phi/\partial x$ at $x = 0$ is obtained from the boundary condition (12) and the solution of (23) is given by

$$\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z) = C_1(\alpha)\exp(\gamma z) + C_2\exp(-\gamma z)$$
$$+ 2i\alpha_0 A_0 \sinh\gamma_0 z/(2\pi)^{1/2}(\alpha^2 - \alpha_0^2). \tag{24}$$

To determine C_1 and C_2, the boundary condition (11), i.e.,

$$\phi_t = 0 \quad \text{or} \quad (\phi)_{z=0} = -(\phi_i)_{z=0} = 0 \tag{25}$$

is used to get

$$\bar{\phi}_-(\alpha, 0) = 0. \tag{26}$$

Changing α to $-\alpha$ in (26) and adding the new equation to it, we have

$$\bar{\phi}_-(\alpha, 0) + \bar{\phi}_-(-\alpha, 0) = 0. \tag{27}$$

Putting $z = 0$ in (24) and using (27), we see that $C_1 = -C_2$, then (24) becomes

$$\bar{\phi}_-(\alpha, z) + \bar{\phi}_-(-\alpha, z) = 2C_1\sinh\gamma z + 2i\alpha_0 A_0 \sinh\gamma_0 z/(2\pi)^{1/2}(\alpha^2 - \alpha_0^2),$$
$$0 \leqslant z \leqslant H. \tag{28}$$

Eliminating C_1 between (28) and its derivative w.r.t. z after putting $z = H$, we find

$$\bar{\phi}_-(\alpha) + \bar{\phi}_-(-\alpha) = \gamma^{-1}\tanh\gamma H[\bar{\phi}_-(\alpha) + \bar{\phi}_-(-\alpha)] + 2i\alpha_0 A_0 \sinh\gamma_0 H/$$
$$(2\pi)^{1/2}(\alpha^2 - \alpha_0^2) - 2i\alpha_0 A_0\gamma_0 \cosh\gamma_0 H \tanh\gamma H/(2\pi)^{1/2}\gamma(\alpha^2 - \alpha_0^2). \tag{29}$$

Integrating (3) again as x varies from l to ∞, after multiplying it by $\exp(i\alpha x)/(2\pi)^{1/2}$ and noting that ϕ and its derivative w.r.t. x vanish as $x \to \infty$, we obtain

$$d^2[\bar{\phi}_{+l}(\alpha, z)]/dz^2 - \gamma^2[\bar{\phi}_{+l}(\alpha, z)] = \exp(i\alpha l)[(\partial\phi/\partial x)_l - i\alpha(\phi)_l]/(2\pi)^{1/2}. \tag{30}$$

Changing α to $-\alpha$ in (30) and adding the new result to it, we have

$$(d^2/dz^2 - \gamma^2)[\bar{\phi}_{+l}(\alpha, z)\exp(-i\alpha l) + \bar{\phi}_{+l}(-\alpha, z)\exp(i\alpha l)]$$
$$= 2(\partial\phi/\partial x)_l/(2\pi)^{1/2}. \tag{31}$$

Using (12), i.e.,

in (31), we obtain the solution of (31) to be

$$\bar{\phi}_{+l}(\alpha, z)\exp(-i\alpha l) + \bar{\phi}_{+l}(-\alpha, z)\exp(i\alpha l)$$

$$= B_1(\alpha)\exp(\gamma z) + B_2(\alpha)\exp(-\gamma z)$$

$$- 2iA_0\alpha_0 \sinh\gamma_0 z \exp(-i\alpha_0 l)/(2\pi)^{1/2}(\alpha^2 - \alpha_0^2). \tag{33}$$

Eliminating B_1 and B_2 by the procedure for C_1 and C_2, we obtain

$$\bar{\phi}_{+l}(\alpha)\exp(-i\alpha l) + \bar{\phi}_{+l}(-\alpha)\exp(i\alpha l) = -2iA_0\alpha_0 \sinh\gamma_0 H \exp(-i\alpha_0 l)/$$

$$(2\pi)^{1/2}(\alpha^2 - \alpha_0^2) + \tanh\gamma H[\bar{\phi}_{+l}(\alpha)\exp(-i\alpha l) + \bar{\phi}_{+l}(-\alpha)\exp(i\alpha l)]/\gamma$$

$$+ \tanh\gamma H[2i\alpha_0\gamma_0 A_0 \cosh\gamma_0 H \exp(-i\alpha_0 l)]/(2\pi)^{1/2}\gamma(\alpha^2 - \alpha_0^2). \tag{34}$$

We now multiply (3) by $\exp(i\alpha x)/(2\pi)^{1/2}$ and integrate it between the limits $x = 0$ to $x = l$, on the line $z = H$, to obtain

$$(d^2/dz^2 - \gamma^2)[\bar{\phi}_l(\alpha, z)] = -\exp(i\alpha l)(\partial\phi/\partial x)_l + (\partial\phi/\partial x)_0$$

$$+ i\alpha\exp(i\alpha l)(\phi)_l - i\alpha(\phi)_0. \tag{35}$$

Using boundary condition (12) and the fact that $\phi_t = 0$, at $(0, H)$ and (l, H), (35) becomes

$$(d^2/dz^2 - \gamma^2)[\bar{\phi}_l(\alpha, z)]$$

$$= -iA_0(\alpha + \alpha_0)\sinh\gamma_0 z[\exp i(\alpha - \alpha_0)l - 1]/(2\pi)^{1/2}, \quad z = H. \tag{36}$$

Its solution is

$$\bar{\phi}_l(\alpha, z) = E_1\exp(\gamma z) + E_2\exp(-\gamma z) + iA_0\sinh\gamma_o z$$

$$\times [\exp i(\alpha - \alpha_0)l - 1]/(2\pi)^{1/2}(\alpha - \alpha_0) \tag{37}$$

Since $\bar{\phi}_l(\alpha, z) \to 0$ as $\alpha \to \infty$, therefore $E_1 = 0$, thus

$$\bar{\phi}_l(\alpha, z) = E_2\exp(-\gamma z)$$

$$+ iA_0\sinh\gamma_0 z[\exp i(\alpha - \alpha_0)l - 1]/(2\pi)^{1/2}(\alpha - \alpha_0). \tag{38}$$

Further, the condition (13), i.e., for $0 \leqslant x \leqslant l$

$$(\partial\phi_t/\partial z)_{z=H} = 0 \quad \text{or} \quad (\partial\phi/\partial z)_{z=H} = -(\partial\phi_i/\partial z)_{z=H}$$

$$= -A_0\gamma_0 \cosh\gamma_o H \exp(-i\alpha_0 x) \tag{39}$$

when integrated between the limits $x = 0$ to $x = l$, after multiplying it by

$$(2\pi)^{1/2}(\alpha^2-\alpha_0^2)-iA_0\sinh\gamma_0H[(\alpha+\alpha_0)\exp(-i\alpha l)-(\alpha-\alpha_0)\exp(i\alpha l)]/$$

$$(2\pi)^{1/2}(\alpha^2-\alpha_0^2). \tag{41}$$

Adding (34) and (41), we have

$$\bar{\phi}_+(\alpha)\exp(-i\alpha l)+\bar{\phi}_+(-\alpha)\exp(i\alpha l)=\tanh\gamma H[\bar{\phi}'_{+l}(\alpha)\exp(-i\alpha l)$$

$$+\bar{\phi}'_{+l}(-\alpha)\exp(i\alpha l)]/\gamma+\tanh\gamma H\,2i\alpha_0\gamma_0A_0\cosh\gamma_0H$$

$$\times\exp(-i\alpha_0 l)/(2\pi)^{1/2}(\alpha^2-\alpha_0^2)-iA_0\sinh\gamma_0H[(\alpha+\alpha_0)\exp(-i\alpha l)$$

$$-(\alpha-\alpha_0)\exp(i\alpha l)]/(2\pi)^{1/2}(\alpha^2-\alpha_0^2). \tag{42}$$

Using (21), (21a) and (40) in (42), we obtain

$$[\bar{\phi}'_+(\alpha)\exp(-i\alpha l)+\bar{\phi}'_+(-\alpha)\exp(i\alpha l)]\frac{\exp(\gamma H)}{\cosh\gamma H}=\gamma[G(\alpha)\exp(-i\alpha l)$$

$$+G(-\alpha)\exp(i\alpha l)]+\frac{iA_0\gamma_0\cosh\gamma_0H}{(2\pi)^{1/2}}\left(\frac{\exp(i\alpha l)}{\alpha+\alpha_0}-\frac{\exp(-i\alpha l)}{\alpha-\alpha_0}\right)$$

$$\times\frac{\exp(\gamma H)}{\cosh\gamma H}-\frac{iA_0}{(2\pi)^{1/2}}\left(\frac{\exp(i\alpha l)}{\alpha+\alpha_0}-\frac{\exp(-i\alpha l)}{\alpha-\alpha_0}\right)$$

$$\times(\gamma\sinh\gamma_0H+\gamma_0\cosh\gamma_0H). \tag{43}$$

Again using (21) and (21a) in (29), we get

$$[\bar{\phi}'_-(\alpha)+\bar{\phi}'_-(-\alpha)]\frac{\exp(\gamma H)}{\cosh\gamma H}=-\gamma[G(\alpha)+G(-\alpha)]$$

$$-\frac{2i\alpha_0A_0[\gamma\sinh\gamma_0H+\gamma_0\cosh\gamma_0H]}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}$$

$$+\frac{2i\alpha_0A_0\gamma_0\cosh\gamma_0H}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}\frac{\exp(\gamma H)}{\cosh\gamma H}. \tag{44}$$

The functional equations (43) and (44) are solved by the Wiener–Hopf technique. We factorize, as an infinite product

$$L(\alpha)=L_+(\alpha)L_-(\alpha)=\cosh\gamma H\exp(-\gamma H)$$

$$L_\pm(\alpha)=\exp[\mp X(\alpha)-T_\pm(\alpha)]\prod_{n=1}^{\infty}[(1-k^2b_{n-(1/2)}^2)^{1/2}\mp i\alpha b_{n-(1/2)}]$$

$$\times\exp(\pm(i\alpha b_{n-(1/2)})) \tag{45}$$

where $b_{n-(1/2)}=H/(n-\frac{1}{2})\pi$ and $X(\alpha)$ is an arbitrary function of α so chosen as to reduce

$L_{\pm}(\alpha)$ to algebraic polynomials at infinity. The behaviour of $L_-(\alpha)$ as $|\alpha|\to\infty$ is given by

$$L_-(\alpha)\sim\exp[X(\alpha)-T_-(\alpha)]\prod_n[1+i\alpha H/\pi(n-\tfrac{1}{2})]\exp[-i\alpha H/(n-\tfrac{1}{2})\pi]. \tag{46}$$

The asymptotic behaviour of the infinite product in (48) for large α is given by the result (Noble [6] ex. 1.10, p. 41).

$$\prod_n[1+\alpha/(n-\tfrac{1}{2})]\exp[-\alpha/(n-\tfrac{1}{2})]\sim\exp(-C\alpha+\alpha+\tfrac{1}{2})2^{-2\alpha}\alpha^{-\alpha}, \tag{47}$$

for large α, where $C=0{\cdot}5772$ is Euler's constant. Then both $L_+(\alpha)$ and $L_-(\alpha)$ are asymptotic to constants if the arbitrary function $X(\alpha)$ is chosen such that

$$X(\alpha)=-i\alpha H\pi^{-1}(1-C+\log\pi/2kH)+\alpha H/2. \tag{48}$$

We now write (44) as

$$\begin{aligned}&\frac{\bar\phi'_-(\alpha)}{L_-(\alpha)(\alpha^2-k^2)^{1/2}}-\frac{2i\alpha_0\gamma_0A_0\cosh\gamma_0H}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)(\alpha^2-k^2)^{1/2}L_-(\alpha)}=\frac{-\bar\phi'_-(-\alpha)}{L_-(\alpha)(\alpha^2-k^2)^{1/2}}\\&-\frac{[\bar\phi'_+(k)-\bar\phi'_-(-k)]L_+(\alpha)}{(2k)^{1/2}(\alpha-k)^{1/2}}-\frac{[\bar\phi'_+(k)-\bar\phi'_-(-k)]L_+(\alpha)}{(-2k)^{1/2}(\alpha+k)^{1/2}}\\&-\frac{2i\alpha_0A_0\sinh\gamma_0HL_+(\alpha)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)}-\frac{2i\alpha_0A_0\gamma_0\cosh\gamma_0HL_+(\alpha)}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)(\alpha^2-k^2)^{1/2}}.\end{aligned} \tag{49}$$

Equation (49) is decomposed as

$$\begin{aligned}&\frac{1}{(\alpha+k)^{1/2}}\left(\frac{\bar\phi'_-(\alpha)}{L_-(\alpha)(\alpha-k)^{1/2}}-\frac{\bar\phi'_-(-k)}{L_-(-k)(-2k)^{1/2}}\right)\\&-\frac{2i\alpha_0\gamma_0A_0\cosh\gamma_0H}{(2\pi)^{1/2}(\alpha+\alpha_0)(\alpha+k)^{1/2}}\\&\times\left(\frac{1}{L_-(\alpha)(\alpha-\alpha_0)(\alpha-k)^{1/2}}+\frac{1}{iL_-(-\alpha_g)(\alpha_g+\alpha_0)(\alpha_g+k)^{1/2}}\right)\\&+\frac{\bar\phi'_-(-\alpha_i)}{L_-(\alpha)(\alpha-k)^{1/2}(\alpha_i+k)^{1/2}}+\frac{[\bar\phi'_+(k)-\bar\phi'_-(-k)]L_+(k)}{(2k)^{1/2}(\alpha-k)^{1/2}}\\&+\frac{iA_0\sinh\gamma_0HL_+(\alpha_0)}{(2\pi)^{1/2}(\alpha-\alpha_0)}+\frac{2i\alpha_0\gamma_0A_0\cosh\gamma_0HL_+(\alpha_g)}{(2\pi)^{1/2}(\alpha-\alpha_0)(\alpha_g+\alpha_0)(\alpha-k)^{1/2}(\alpha_g+k)^{1/2}}\\&=\frac{1}{L_-(\alpha)(\alpha-k)^{1/2}}\left(\frac{\bar\phi'_-(-\alpha_i)}{(\alpha+k)^{1/2}}-\frac{\bar\phi'_-(-\alpha_i)}{(\alpha_i+k)^{1/2}}\right)\\&\qquad\left[\bar\phi'_+(k)-\bar\phi'_-(-k)\right]\qquad\qquad\left(\bar\phi'_+(k)-\bar\phi'_-(-k)\right)L_+(\alpha)\end{aligned}$$

$$\times\left(\frac{1}{(\alpha+\alpha_0)(\alpha+k)^{1/2}}-\frac{1}{(\alpha_g+\alpha_0)(\alpha_g+k)^{1/2}}\right)-\frac{1}{(\alpha+k)^{1/2}(-2k)^{1/2}L_-(-k)}$$

$$-\frac{2\alpha_0\gamma_0 A_0\cosh\gamma_0 H}{(2\pi)^{1/2}L_-(-\alpha_g)(\alpha_g+\alpha_0)(\alpha+k)^{1/2}(\alpha_g+k)^{1/2}}, \tag{50}$$

where $\alpha_i = k, \alpha_m$ and $\alpha_g = \alpha_0, k; \alpha_m$ are the zeros of $L_-(\alpha) = 0$. The left hand member of (50) is analytic in the region $T < k_2$ and the right hand member in $T > -k_2$. By the argument following (20) each member reduces to zero as $|\alpha| \to \infty$. By equating the left hand member to zero, we get

$$\begin{aligned}\bar{\phi}'_-(\alpha) = {} & \frac{\bar{\phi}'_-(-k)L_-(\alpha)(\alpha-k)^{1/2}}{L_-(-k)(-2k)^{1/2}} + \frac{2i\alpha_0 A_0\gamma_0\cosh\gamma_0 H}{(2\pi)^{1/2}(\alpha^2-\alpha_0^2)} \\ & + \frac{2\alpha_0 A_0\gamma_0\cosh\gamma_0 H(\alpha-k)^{1/2}L_-(\alpha)}{(2\pi)^{1/2}(\alpha+\alpha_0)(\alpha_g+\alpha_0)(\alpha_g+k)^{1/2}L_-(\alpha_g)} - \frac{\bar{\phi}'_-(-\alpha_i)(\alpha+k)^{1/2}}{(\alpha_i+k)^{1/2}} \\ & - \frac{[\bar{\phi}'_+(k)-\bar{\phi}'_-(-k)]L_-(\alpha)L_-(k)(\alpha+k)^{1/2}}{(2k)^{1/2}} \\ & - \frac{iA_0\sinh\gamma_0 HL_+(\alpha_0)L_-(\alpha)\gamma}{(2\pi)^{1/2}(\alpha-\alpha_0)} \\ & - \frac{2i\alpha_0 A_0\gamma_0\cos\gamma_0 HL_+(\alpha_0)L_-(\alpha)(\alpha+k)^{1/2}}{(2\pi)^{1/2}(\alpha-\alpha_0)(\alpha_g+\alpha_0)(\alpha_g+k)^{1/2}}.\end{aligned} \tag{51}$$

In the same manner, factorizing and decomposing (43), we get

$$\begin{aligned}\bar{\phi}'_+(\alpha) = {} & \bar{\phi}'_+(\alpha_i)(\alpha+k)^{1/2}\exp i(\alpha-\alpha_0)l/(\alpha_i+k)^{1/2} \\ & + [\bar{\phi}'_+(k)-\bar{\phi}'_-(-k)\exp(2i\alpha l)]L(\alpha)[(\alpha+k)/(2k)]^{1/2} \\ & - [\bar{\phi}'_+(k)\exp(-i\alpha l)-\bar{\phi}'_-(-k)\exp(ikl)] \\ & \times L_+(k)L_-(\alpha)(\alpha+k)^{1/2}\exp(i\alpha l)/2k \\ & + [\bar{\phi}'_+(k)\exp(2i\alpha l)-\bar{\phi}'_-(-k)]L(\alpha)(\alpha-k)^{1/2}/(-2k)^{1/2} \\ & - iA_0L(\alpha)\exp(2i\alpha l)[\gamma\sinh\gamma_0 H+\gamma_0\cosh\gamma_0 H]/(2\pi)^{1/2}(\alpha+\alpha_0) \\ & + iA_0L(\alpha)[\gamma\sinh\gamma_0 H+\gamma_0\cosh\gamma_0 H]/(2\pi)^{1/2}(\alpha-\alpha_0) \\ & - iA_0L_-(\alpha)\exp(i\alpha l)[\gamma\sinh\gamma_0 HL_+(\alpha_0)\exp(-i\alpha_0 l)(\alpha_g+k)^{1/2} \\ & + \gamma_0\cosh\gamma_0 HL_+(\alpha_g)(\alpha+k)^{1/2} \\ & \times\exp(-i\alpha_g l)]/(2\pi)^{1/2}(\alpha-\alpha_0)(\alpha_g+k)^{1/2} \\ & + iA_0\gamma_0\cosh\gamma_0 HL_+(k)L_-(\alpha)(\alpha+k)^{1/2} \\ & \times\exp i(\alpha+k)l/(2\pi)^{1/2}(2k)^{1/2}(\alpha_0+k) \\ & - \bar{\phi}'_+(k)L_-(\alpha)(\alpha-k)^{1/2}\exp i(\alpha-k)l/(-2k)^{1/2}L_-(-k)\end{aligned}$$

$$\times (\alpha - k)^{1/2} \exp(i\alpha l)[\exp(-i\alpha_g l)/(\alpha + \alpha_0) L_-(-\alpha_g)(\alpha_g + k)^{1/2}$$

$$+ \exp(ikl)/(2k)^{1/2}(k + \alpha_0) L_-(-k)]/(2\pi)^{1/2}. \tag{52}$$

The potential function $\phi(x, z)$ is given by the inverse Fourier transform

$$\phi(x, z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty + iT}^{\infty + iT} \bar{\phi}(\alpha, z) \exp(-i\alpha x)\, d\alpha$$

$$= \frac{-1}{(2\pi)^{1/2}} \int_{-\infty + iT}^{\infty + iT} [\bar{\phi}'(\alpha) \exp(\gamma H - \gamma z - i\alpha x)\, d\alpha]/\gamma, \tag{53}$$

where $\bar{\phi}'(\alpha)$ is obtained by adding (51) and (52) and the integral (53) is carried out along a line in $-k_2 < T < k_2$.

5. Evaluation of the integral

The integral in (53) is evaluated along a closed contour Γ with $\alpha = \pm k$ and $\alpha = \pm \alpha_0$ as the branch points and poles respectively. It also satisfies the condition that $\mathrm{Re}(\gamma) \geqslant 0$. The condition $\mathrm{Re}(\gamma) = 0$ as discussed by Ewing and Press [2] gives the parts of hyperbola to be used as branch cuts with $\alpha = \pm k$ as branch points. We take Γ as shown in figure 2 and the line of integral (53) in $-k_2 < T < k_2$. By Cauchy's residue theorem

$$\int_{-\infty + iT}^{\infty + iT} + \int_A^B + \int_{L\alpha} + \int_C^D = 2\pi i \text{ (sum of residues)}. \tag{54}$$

The factor $\exp(-i\alpha x) = \exp(-i\sigma x)\exp(Tx)$ in the integrand makes the integral along infinite circular arcs AB and CD vanish in the lower part of α-plane when $x > 0$. The residues at the pole $\alpha = \alpha_0$ in the upper part of α-plane are

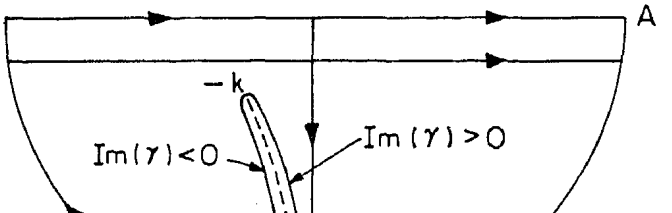
$$-2A_0 \cosh \gamma_0 H \exp(-\gamma_0 H + \gamma_0 z - i\alpha_0 x), \quad \gamma(\alpha_0) = -\gamma_0, \quad \alpha_g = \alpha_0 \tag{55a}$$

and

$$A_0 \cosh \gamma_0 H \left(1 - \frac{2\alpha_0 \exp(-\gamma_0 H) \cosh \gamma_0 H}{(2k)^{1/2}(\alpha_0 + k)^{1/2}} + \exp(-2\gamma_0 H) \right.$$

$$\left. - \frac{L_-(-\alpha_0) L_+(k)(\alpha_0 + k)^{1/2}}{(2k)^{1/2}} \right) \exp(\gamma_0 H - \gamma_0 z - i\alpha_0 x),$$

$$\gamma(\alpha_0) = \gamma_0, \quad \alpha_g = k. \tag{55b}$$



where

$$A_0' = A_0 \cosh \gamma_0 H(1 + \exp(-2i\alpha_0 l) - L_-(-\alpha_0)$$

$$\times [2\alpha_0 + (\alpha_0 + k)\exp[-i(\alpha + k)l]])/(2k)^{1/2} L_-(-k)(\alpha_0 + k)^{1/2}. \tag{55e}$$

We now consider

$$I = \frac{-1}{(2\pi)^{1/2}} \int_{L\alpha} [\bar{\phi}'(\alpha) \exp(\gamma H - \gamma z - i\alpha x)]\, d\alpha]/\gamma \tag{56}$$

where $L\alpha$ is the branch cut. On $L\alpha$, $\mathrm{Re}(\gamma) = 0$ and it is seen that $\mathrm{Im}(\gamma)$ has different signs on the two sides of the branch cut as shown in figure 2. As pointed out earlier, the factor $\exp(Tx)$ gives exponential decrease of the integrand and makes it vanish at $T = -\infty$. Therefore the main contribution comes from the neighbourhood of the branch point $\alpha = -k$. For this, we set $\alpha = -k - iu$ such that $\gamma^2 = \alpha^2 - k^2 = 2ik_1 u - 2k_2 u - u^2$. Since γ is imaginary and so γ^2 is real and negative. Therefore $\gamma \to \pm i(2k_2 u + u^2)^{1/2} = \pm i\gamma'$ for small u. Using this value of γ in (56), we have

$$I = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{\bar{\phi}'(\alpha) \exp(\gamma H - \gamma z)}{\gamma}\bigg|_{\gamma = i\gamma'} \exp(ikx - ux)(-idu)$$

$$- \frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{\bar{\phi}'(\alpha) \exp(\gamma H - \gamma z)}{\gamma}\bigg|_{\gamma = -i\gamma'} \exp(ikx - ux)(-idu). \tag{57}$$

The value of $\bar{\phi}'(\alpha)$ is obtained from (51) and (52), then (57) becomes

$$I = \exp(ikx) \int_0^\infty \bigg[M_1 \frac{\cos\gamma'(z - H) - \cos\gamma'(z + H)}{\gamma'}$$

$$+ M_2 \sin\gamma'(z + H) + M_3 \sin\gamma'(z - H) \bigg] \exp(-ux)\, du, \tag{58}$$

where

$$M_1 = \frac{-1}{(2\pi)^{1/2}} [(\bar{\phi}'_-(-k) - \bar{\phi}'_+(k)) \exp(-2ikl)]$$

$$+ \frac{iA_0\gamma_0 \cosh\gamma_0 H}{(2\pi)^{1/2}} \left[\frac{\exp(-2ikl)}{-k + \alpha_0} + \frac{1}{k + \alpha_0} \right]$$

$$M_2 = \frac{iA_0 \sinh\gamma_0 H}{2\pi} \left[\frac{\exp(-2ikl)}{-k + \alpha_0} + \frac{1}{k + \alpha_0} \right]$$

$$M_3 = \frac{iA_0 \sinh\gamma_0 H}{2\pi} \left[\frac{\exp(-2ikl)}{-k + \alpha_0} + \frac{1 - 4L_+(\alpha_0)L_-(-k)}{k + \alpha_0} \right]. \tag{59}$$

The values of M_i are obtained from (57) by putting $\alpha = -k$ in the coefficients of trigonometrical functions of z. The integrals in (58) are Laplace integrals. To evaluate these integrals, we use results by Oberhettinger (1973, p. 69–70) [7], i.e.,

$$\exp(-k_2x)\int_0^\infty \frac{\cos(2k_2u+u^2)^{1/2}z}{(2k_2u+u^2)^{1/2}}\exp(-ux)\,du = K_0(k_2r) \tag{60a}$$

and

$$\exp(-k_2x)\int_0^\infty \sin(2k_2u+u^2)^{1/2}z\exp(-ux)\,du = \frac{k_2z}{(r)^{1/2}}K_1(k_2r) \tag{60b}$$

where $K_n(x)$ are the modified Hankel functions of order n and

$$r = (x^2+z^2)^{1/2} \tag{61}$$

$$K_n(x) = \frac{\pi}{2}i^{n+1}[J_n(ix)+i\gamma_n(ix)]. \tag{62}$$

Further, the values of $J_n(X)$ and $\gamma_n(X)$ are given by

$$J_n(X) = \sum_{r=0}^{\infty}\frac{(-1)^r}{\lfloor r\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r} \tag{63}$$

and

$$\gamma_n(X) = \frac{2}{\pi}\left[C+\log\left(\frac{x}{2}\right)\right]J_n(X) - \sum_{k=0}^{n-1}\frac{(n-k-1)x^{-n+2k}}{2^{-n+2k}\lfloor k} - \sum_{k=0}^{\infty}\frac{(-1)^k x^{n+2k}}{2^{n+2k}\lfloor k\lfloor n+k}[\psi(n+k)+\psi(k)]. \tag{64}$$

Here C is Euler's constant, $\psi(0)=0$ and

$$\psi(k) = 1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\frac{1}{k}. \tag{65}$$

Using (60a) and (60b) in (58) and evaluating other terms around $u=0$, we obtain

$$I(x,z) = \left[M_1[K_0(k_2r_1)-K_0(k_2r_2)] + \frac{M_2k_2[z+H]}{(r_2)^{1/2}}K_1(k_2r_2) + \frac{M_3k_2(z-H)K_1(k_2r_1)}{(r_1)^{1/2}}\right]\exp(-k_2l), \tag{66}$$

where

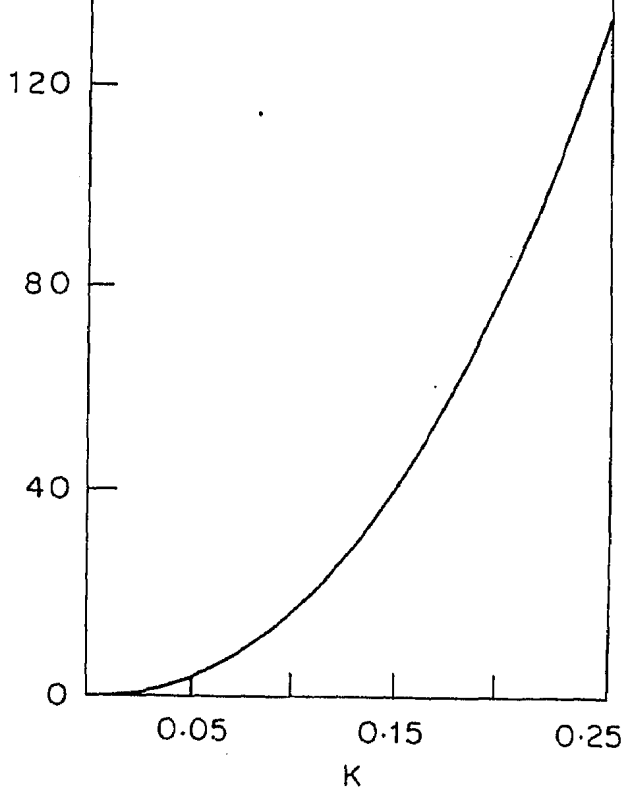


Figure 3. Amplitude of the scattered wave vs the wave number.

vertical barrier and

$$I(x,z) = \bar{M}_1[K_0(k_2r_1) - K_0(k_2r_2)] + \frac{\bar{M}_2 k_2(z+H)K_1(k_2r_2)}{(r_2)^{1/2}}$$

$$+ \frac{\bar{M}_3 k_2(z-H)K_1(k_2r_1)}{(r_1)^{1/2}}$$

where $\bar{M}_i$ are the values of M_i when $l = 0$.

6. Conclusions

The waves reflected from the boundary $z = H$, and transmitted to the region $x < 0$ are given by (55a) and (55b) respectively. In (55d), we have the waves which are reflected from the boundary $x = l$ of the strip. The waves reflected from the strip and then from the free surface of the ocean in the region $x > 0$ are obtained in (55c). The scattered waves in (67) are obtained in terms of Bessel functions which have a known behaviour for small and large values of $r = ((x-l)^2 + (z-H)^2)^{1/2}$. The scattered field is known at points close and far away from the scatterer. The numerical calculations for the amplitude of the scattered waves have been obtained for $r = 10$ km, $H = 1$ km and $l = 1$ km. The amplitude increases rapidly as the wave number increases very slowly. The scattered waves originate as cylindrical waves at the corner (l, H) of the strip and at its image $(l, -H)$ in the free surface. They die out as they move away from the scatterer.

l ($z = 0$, $0 \leqslant x \leqslant l$) in the free surface of the ocean. In case the strip is a thin plane vertical barrier ($x = 0, 0 \leqslant z \leqslant H$), the scattered waves are given by (69). In figure 3, the amplitude of the scattered wave is plotted as a function of the wave number.

The paper has its application to scattering of seismic Rayleigh waves due to the presence of submerged ice in oceans. The results are valid even at distant points. With the spread of seismic stations all over the world, the results of the present paper can be used to study the scattering of seismic waves even in remote places like Antarctica.

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Sur le rang de la jacobienne des systèmes linéaires de quadriques

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Abstract. A necessary and sufficient condition is shown in order that a linear system of quadrics in S_r, joined of first and second kind, has a Jacobian with rank $r-k$.

Keywords. Quadrics; linear system of quadrics.

1. Introduction

Le problème de déterminer tous les systèmes linéaires de quadriques à Jacobienne identiquement nulle $L_{d/m}$, où d est la dimension du système linéaire et $m \leqslant d$ est le rang de la Jacobienne a été affronté pour la première fois par Bonferroni (1914) [2], qui a determiné tous les systèmes de ce type en S_3.

Successivement Terracini (1919) [7], a mis en évidence l'importance de ce problème avec les suivantes considérations.

Donnée une variété à k dimensions V_k, plongée dans un espace projectif $S_r (r \geqslant 2k)$, elle possède en général ∞^k espaces S_k tangents, qui couvrent une variété W de dimension $2k$. I1 peut arriver, pour des V_k particulières, que la rélative W ait dimension inférieure à $2k$.

Terracini a démontré que:

"*Si la variété W, rélative à une* V_k, *a dimension* $2k-s$, *la* V_k *satisfait à*:

$$d = \tfrac{1}{2}k(k-1) + s$$

équations de Laplace linéairement indépendantes ou bien à un système d'équations de Laplace parmi les quelles moins de d sont linéairement indépendantes et pour cela telles que le système des formes quadratiques associées a un système apolaire à Jacobienne de rang $k-s$ *et viceversa.*

Pour cela il surgit la question de determiner tous les systèmes linéaires de formes quadratiques à k variables (c'est à dire systèmes linéaires de quadriques de S_{k-1}), dont la matrice Jacobienne a rang $k-s$.

Muracchini (1953) [6] détermina tous les systèmes linéaires de quadriques de ce type en S_4.

Les variétés-bases des systèmes déterminés par cet auteur sont constitutées par des espaces linéaires.

En partant d'un travail de Xambò [8] nous avons démontré (1983–85) (voir: [3], [4], et [5]), que à partir de S_5 il existe des systèmes linéaires de quadriques à Jacobienne

identiquement nulle qui ont pour bases des $S_i - V^{r-1}_{1+i}$ et des variétés de Sergre du même type.

Maintenant nous cherchons de donner une contribution définitive à cette question en S_r.

2. Dans l'éspace linéaire complexe S_r de coordonnées projectives homogènes $x_i (i = 0, 1, \ldots, r)$ choisissons $d + 1$ quadriques linéairement indépendantes:

$$f_0 = 0, \quad f_1 = 0, \ldots, f_d = 0$$

avec:

$$f_q = \sum_{i,k=0}^{r} a_q^{ik} x_i x_k \quad (a_q^{ik} = a_q^{ki}).$$

Le système linéaire $L_{d/m}$ de dimension d et Jacobienne de rang m est exprimé par l'équation:

$$\sum_{q=0}^{d} \rho_q f_q = 0.$$

Supposons que la matrice Jacobienne à $r + 1$ lignes et $d + 1$ colonnes:

$$J = \left\| \frac{\partial f_q}{\partial x_i} \right\| \begin{pmatrix} q = 0, 1, \ldots, d \\ i = 0, 1, \ldots, r \end{pmatrix}$$

soit à rang $m \leqslant d$.

Souvent si $m \leqslant r$ nous mettons: $m = r - k$ et le système sera indiqué evidemment avec: $L_{d/r-k}$.

Quand la matrice Jacobienne de L_d en S_r est identiquement nulle de rang r les S_{r-1} polaires d'un point générique de S_r passent tous par un point P' (et non par un espace S_g avec $g > 0$).

On obtient ainsi dans S_r une correspondance T dans la quelle à P il correspond P' réciproquement pour la lois de réciprocité des polaires. Cette correspondance sera en général biunivoque ou bien elle poura être telle que aux points de S_r il correspond tous les points d'un espace linéaire.

Si la matrice Jacobienne à rang $r - h$ un point générique P a pour conjugué un S_h.

Nous dirons qu'un système de quadriques $L_{d/m}$ est "*déjoint*" quand ils existent des systèmes subordonnés sans quadriques en commun:

$$L_{d_1/m_1}, L_{d_2/m_2}, \ldots, L_{d_s/m_s}$$

et éventuellement p $(p \geqslant 0)$ quadriques fonctionellement indépendantes, de manière qu'ils soient satisfaites les égalités:

$$d = d_1 + d_2 + \cdots + d_s + s + p - 1,$$

$$m = m_1 + m_2 + \cdots + m_s + p.$$

Autrement dît il sera nommé: "*conjoint*".

Un système conjoint $L_{d/m}$ possède toujours des systèmes subordonnés "banaux"

linéaire $L_{m/m}$.

Mais on ne dit pas que $L_{d/m}$ possède toujours les systèmes subordonnés "essentiels" $L_{g/c}$ avec:

$$2 \leqslant g \leqslant d-1, \quad 2 \leqslant c \leqslant m-1, \quad c < g.$$

Lorsque ces dernièrs systèmes existent, ils imposent à $c+1$ quadriques linéairement indépendantes, choisies dans $L_{g/c}$ d'être fonctionellement dépendantes. Nous dirons systèmes: "*conjoints de première espèce*" les systèmes, qui ne possèdent pas des systèmes subordonnés essentiels, et systèmes "*conjoints de seconde éspèce*" les systèmes qui, n'etant pas déjoints, possèdent toutefois des systèmes subordonnés essentiels.

Soient: $L_{d_1}, L_{d_2}, \ldots, L_{d_s}$ les systèmes essentiels contenus dans $L_{d/m}$. Ils devront former une "chaine", c'est à dire:

1°) aucun d'eux est déjoint,

2°) L_{d_1} a au moins une quadrique en commun avec un autre, par exemple L_{d_2}, et leur système-union L_a a au moins une quadrique en commun avec un troisième, par exemple L_{d_3}, et le système-union de L_a avec L_{d_3} a au moins une quadrique en commun avec un quatrième système et toujours ainsi jusqu'a épuiser tout $L_{d/m}$.

Dans L_d il pourra exister plus qu'une chaine.

Pour prouver cette assertion démontrons le:

LEMME 1. *Un système $L_{d/m}$, qui ne soit pas déjoint et pas même conjoint de première espèce est certainement conjoint de seconde espèce et possède au moins un chaine de systèmes subordonnés.*

Démonstration

Considérons un système $L_{d/m}$, qui ne soit pas déjoint et pas même conjoint de prémière espèce. Alors il possèdera au moins un système subordonné essentiel L_{d_1/m_1} $(m_1 \leqslant d_1)$ avec $m_1 < m$.

Considérons les $d-d_1$ quadriques restantes, qui forment un système $L_{d-d_1-1/p}$ et observous che p ne peut pas être $< m-m_1$, autrement dit le nombre des quadriques fonctionellement indépendantes dans $L_{d/m}$ serait: $m_1+p < m_1+(m-m_1)$, c'est à dire: $m_1+p < m$, tandis qu'elles sont: m.

Il ne peut pas être $p = m-m_1$ car les deux systèmes subordonnés L_{d_1/m_1} et $L_{d-d_1-1/p}$, qui n'ont pas des quadriques en commun, satisferaient aux égalités:

$$d_1+(d-d_1-1)+1 = d$$

$$m_1+p = m.$$

Pour cela $L_{d/m}$ serait déjoint contre l'hypothèse.

Donc il est: $p > m-m_1$. Cela signifie que quelque quadrique parmi les $d-d_1$ restantes a un lien fonctionel avec au moins une quadrique de L_{d_1/m_1} sinon le nombre des quadriques fonctionellement indépendantes de $L_{d/m}$, serait $> m$, ce qui est impossible.

Donc il existe un système L_{d_2/m_2} qui a des quadriques en commun avec L_{d_1/m_1}.

Indiquons avec $L_{a/n}$ le système-union de L_{d_1/m_1} et L_{d_2/m_2} et considerons les $d-a$

$L_{a/m}$, tandis que les autres apparterrons à $L_{d-a-1/q}$. Ainsi poursuivant on arrive à épuiser tout $L_{d/m}$.

Donc $L_{d/m}$ est un système conjoint de seconde espèce dans le quel:

$$L_{d_1/m_1}, L_{d_2/m_2}, L_{d_3/m_3}, \ldots\ldots$$

forment une chaine.

LEMME 2. *Si le système L_d possède les systèmes subordonnés: $L_{d_1}, L_{d_2}, \ldots, L_{d_s}$ qui forment une chaine il est toujours possible choisir parmi les quadriques de ces systèmes, qui passent par un point générique P de S_r, d quadriques linéairement indépendantes de manière d'individualiser le système L_{d-1} de quadriques de L_d, qui passent par P.*

Démonstration

Nous pouvons écrire les équations de L_{d_1} et L_{d_2} ainsi:

$$\begin{aligned} &\rho_0 f_0 + \rho_1 f_1 + \cdots + \rho_h f_h + \cdots + \rho_{d_1} f_{d_1} = 0, \\ &\sigma_0 g_0 + \sigma_1 g_1 + \cdots + \sigma_h g_h + \cdots + \sigma_{d_2} g_{d_2} = 0. \end{aligned} \tag{1}$$

L'équation due système-union L_a résulte:

$$\begin{aligned} &\nu_0 f_0 + \nu_1 f_1 + \cdots + \nu_h f_h + \cdots + \nu_{d_1} f_{d_1} + \tau_0 g_0 \\ &+ \tau_1 g_1 + \cdots + \tau_h g_h + \cdots + \tau_{d_2} g_{d_2} = 0. \end{aligned} \tag{2}$$

Puisque L_{d_1} et L_{d_2} ont en commun au moins une quadrique, nous pouvons supposer que f_h coincide avec g_h. Soit P un point de S_r.

Notons avec: $f_0(P), f_1(P), \ldots, g_0(P), g_1(P), \ldots\ldots$ les valeurs des quadriques en P. En remplaçant les coordonnées de P dans (1) on peut trouver les valeurs de ρ_h et σ_h en fonction des ρ_i et σ_i restants, après quoi les deux systèmes deviennent:

$$\begin{aligned} &\rho_0\left(f_0 - \frac{f_0(P)}{f_h(P)} f_h\right) + \rho_1\left(f_1 - \frac{f_1(P)}{f_h(P)} f_h\right) + \cdots \\ &+ \rho_{d_1}\left(f_{d_1} - \frac{f_{d_1}(P)}{f_h(P)} f_h\right) = 0, \\ &\sigma_0\left(g_0 - \frac{g_0(P)}{f_h(P)} f_h\right) + \sigma_1\left(g_1 - \frac{g_1(P)}{f_h(P)} f_h\right) + \cdots \\ &+ \sigma_{d_2}\left(g_{d_2} - \frac{g_{d_2}(P)}{f_h(P)} f_h\right) = 0. \end{aligned} \tag{3}$$

Ces systèmes manquent par rapport aux prémiers des paramètres ρ_h et σ_h.

Ils résultent les systèmes L_{d_1-1}, L_{d_2-1} de quadriques de L_{d_1} et L_{d_2}, qui passent par

$$v_0\left(f_0-\frac{f_0(P)}{f_h(P)}f_h\right)+v_1\left(f_1-\frac{f_1(P)}{f_h(P)}f_h\right)+\cdots+v_{d_1}\left(f_{d_1}-\frac{f_{d_1}(P)}{f_h(P)}f_h\right)$$

$$+\tau_0\left(g_0-\frac{g_0(P)}{f_h(P)}f_h\right)+\tau_1\left(g_1-\frac{g_1(P)}{f_h(P)}f_h\right)+\cdots \tag{4}$$

$$+\tau_{d_2}\left(g_{d_2}-\frac{g_{d_2}(P)}{f_h(P)}f_h\right)=0.$$

Puisque le système linéaire (4) est individualisé par les mêmes quadriques linéairement indépendantes des systèmes (3), il en résulte que le système L_{d-1} des quadriques de L_d, qui passent par P, est individualisé par d quadriques linéairement independantes de L_{d_1-1} et L_{d_2-1}.

Mais par hypothèse les systèmes subordonnés de L_a forment une chaine.

Donc le système-union L_a possède en commun avec L_{d_3} au moins une quadrique. En opérant de même façon on prouvera que le système L_{b-1} de quadriques de L_b, système-union de L_a avec L_{d_3}, qui passent par P, est individualisé par b quadriques linéairement indépendantes de L_{a-1} et L_{d_3-1}, c'est à dire de L_{d_1-1}, L_{d_2-1}, L_{d_3-1}.

En continuant de cette manière le lemma résulte démontré.

2. Théorème

Une condition necessaire et suffisante pour que le système linéaire conjoint de quadriques L_d soit à Jacobienne de rang $r-k(k\geqslant 0)$ est que les quadriques du système qui passent par un point quelconque de S_r possèdent en commun un S_{k+1}.

Démontrons avant-tout la suffisance.

Si toutes les quadriques d'un système linéaire L_d de S_r, qui passent par un point générique P, ont en commun un S_{k+1} par P il est évident que le point P a pour conjugué le même S_{k+1} par rapport à toutes les quadriques du système L_{d-1}, qui passent par P. Un'autre quadrique de L_d, qui ne passe pas par P, ne contient pas S_{k+1}, le quel n'est pas contenu dans l'hyperplan polaire de P par rapport à cette quadrique, autrement dit P serait contenu dans la quadrique.

Donc l'hyperplan coupera S_{k+1} dans un S_k et pour cela le point P a pour conjugué un S_k par rapport à toutes les quadriques du système L_d. Cela signifie que la Jacobienne a rang $r-k$, comme il fallait démontrer.

Démontrons maintenant la nécessité.

I) Formons l'hypothèse que le système soit conjoint de prémière espèce et supposons qu'il soit: $r\leqslant d$.

Considérons avant-tout le cas particulier $k=0$ et démontrons que: "*Si le système L_d est à Jacobienne de rang r les quadriques du système qui passent par un point ont en commun une droite*".

Si la Jacobienne est de rang r, cela signifie que tous les déterminants de la Jacobienne d'ordre $r+1$ sont identiquement nuls.

Considerons le déterminant individualisé par $r+1$ quadriques quelconques.

Nous pouvons choisir les premiers $r+1$ quadriques du système et on aura:

$$D = \left|\frac{\partial f_i}{\partial x_s}\right| \quad \begin{pmatrix} i = 0, 1, \ldots, r \\ s = 0, 1, \ldots, r \end{pmatrix}. \tag{5}$$

Les mineurs d'ordre r extraits d'une quelconque matrice constituée par r colonnes du déterminant D ne sont pas tous nuls, sinon il existerait dans $L_{d/r}$ le système subordonné essentiel $L_{r-1/r-1}$, contre l'hypothèse que L_d soit conjoint de prémièr espèce.

Il est donc nécessaire qu'un au moins de ces mineurs soit $\neq 0$. Nous pouvons supposer qu'il soit le mineur obtenu en éliminant la dernière ligne et la dernière colonne de D. Nous l'indiquérons par A_r.

Prénons en considération la matrice extraite de D, constituée avec les prémières r lignes et indiquons par $A_0, A_1, \ldots, A_{r-1}$ les mineurs d'ordre r qu'on obtient en remplaçant à la prémière, à la seconde, ect. colonne de A_r la dernière colonne de la matrice changée de signe.

Puisque le déterminant D est identiquement nul les $r+1$ quadriques sont fonctionellement dépendantes. On aura:

$$F(f_0, f_1, \ldots, f_r) = 0. \tag{6}$$

Cette relation est exacte pour tous les groupes de $r+1$ quadriques choisies entre L_d, mais il n'est pas possible que des quadriques en nombre $< r+1$ soient fonctionellement dépendantes entre elles, autrement dit il existerait dans L_d des systèmes subordonnés essentiels contre l'hypothèse. Pour cela un'égalité analogue à (6) est impossible si le nombre des quadriques est $< r+1$. En dérivant (6) on obtient:

$$\sum_{i=0}^{r} \frac{\partial F}{\partial f_i}\frac{\partial f_i}{\partial x_k} = 0 \quad (k = 0, 1, \ldots, r) \tag{7}$$

qui est un système algébrique de prémièr dégré par rapport aux $\partial F/\partial f_i$ dans le quel le déterminant principal est nul.

Donc le dernière équation résulte une combinaison linéaire des précédentes. En éliminant cette équation et en résolvant le système des restantes équations par rapport à: $(\partial F/\partial f_i)/(\partial F/\partial f_r)$ on obtient:

$$\frac{\partial F}{\partial f_i}\Big/\frac{\partial F}{\partial f_r} = -\frac{A_i}{A_r} \quad (i = 0, 1, \ldots, r-1). \tag{8}$$

Considérons un point x de S_r de coordonnées $x_0, x_1, \ldots, x_r$ et soit $x'(x'_0, x'_1, \ldots, x'_r)$ son conjugué par rapport à toutes les quadriques du système. La droite qui joint les deux points sera donnée par:

$$y_i = t_1 x_i + t_2 x'_i \quad (i = 0, 1, \ldots, r). \tag{9}$$

En remplaçant (9) dans toutes les quadriques on obtient pour la quadrique générique f_m:

çant (10) dans (6) et aussitôt en dérivant par rapport à t_1 et t_2 on obtient:

$$\frac{\partial f_r}{\partial t_1} = \sum_{s=0}^{r-1} \frac{\partial F}{\partial f_s} \frac{\partial f_s}{\partial t_1}, \quad \frac{\partial f_r}{\partial t_2} = \sum_{s=0}^{r-1} \frac{\partial F}{\partial f_s} \frac{\partial f_s}{\partial t_2}. \tag{11}$$

En dérivant (10) on a:

$$\frac{\partial f_m}{\partial t_1} = 2t_1 f_m(x), \quad \frac{\partial f_m}{\partial t_2} = 2t_2 f_m(x') \quad (m = 0, 1, \ldots, r).$$

En remplaçant dans (11):

$$\sum_{s=0}^{r} \frac{\partial F}{\partial f_s} f_s(x) = 0, \quad \sum_{s=0}^{r} \frac{\partial F}{\partial f_s} f_s(x') = 0$$

et enfin pour (8):

$$\sum_{i=0}^{r} A_i f_i(x) = 0, \quad \sum_{i=0}^{r} A_i f_i(x') = 0. \tag{12}$$

Maintenant considérons le système linéaire des quadriques de L_r qui contiennent la droite xx'. Il s'agit d'un système de dimension $r-2$, c'est à dire d'un système linéaire individualisé par $r-1$ quadriques linéairement indépendantes.

Sans contredire la généralité, nous pouvons supposer que ces quadriques sont $r-1$ quadriques choisies parmi les $r+1$ quadriques linéairement indépendantes: $f_0, f_1, \ldots, f_r$ et que les deux quadriques restantes, qui ne contiennent pas la droite xx', soient f_h et $f_k (h < k)$.

On aura:

$$\begin{matrix} f_i(x) = 0 \\ f_i(x') = 0 \end{matrix} \quad (i = 0, 1, \ldots, h-1, h+1, \ldots, k-1, k+1, \ldots, r). \tag{13}$$

Les (12) deviennent:

$$A_h f_h(x) + A_k f_k(x) = 0$$

$$A_h f_h(x') + A_k f_k(x') = 0$$

et pour cela:

$$\frac{f_h(x)}{f_k(x)} = \frac{f_h(x')}{f_k(x')}. \tag{14}$$

Puisque f_h et f_k son des quadriques quelconques, il s'ensuit que en général il sera:

$$f_p(x) = c f_p(x') \tag{15}$$

où f_p est une quadrique générique de L_d et c une costante pas nulle.

A' vrai dire dans les cas qu'un des A_i $(i = 0, 1, \ldots, r)$, par exemple A_i $(0 \leq i \leq r)$ était

nul, il ne serait pas possible pour la seule quadrique f_j déduire que:

$$f_j(x) = cf_j(x')$$

Mais en ce cas, considérons toutes les r quadriques, dont les dérivées partielles paraissent en A_j et la matrice formée avec les colonnes du déterminant D dans les quelles ces quadriques paraissent.

Nous savons que au moins un déterminant de cette matrice n'est pas nul.

Notons ce dernier avec A'_j et répétons la démonstration en remplaçant A_r avec A'_j. Nous obtenons ainsi les déterminants:

$$A'_0, A'_1, \ldots, A'_{r-1}, A'_r = A'_j$$

et en répetant toutes les argumentations précédentes, nous parvenons à des conclusions analogues, c'est à dire aux formules:

$$\begin{matrix} \sum_{i=0}^{r} A'_i f_i(x) = 0 \\ \sum_{i=0}^{r} A'_i f_i(x') = 0 \end{matrix} \qquad (i = 0, 1, \ldots, r) \tag{16}$$

des quelles nous tirerons les mêmeç conclusions.

Mais cette fois la quadrique f_j satisfait à (15) parce que A'_j n'est pas nul.

De cette manière la démonstration n'a pas des exceptions.

On en déduit que toutes les quadriques du système L_d qui passent par un point x passent aussi par son conjugué x' et réciproquement. Si x est situé sur la quadrique f_p, on aura:

$$f_p(x) = 0$$

et pour (15):

$$f_p(x') = 0.$$

Il s'ensuit:

$$t_1^2 f_p(x) + t_2^2 f_p(x') = 0$$

et pour (10):

$$f_p(y) = 0$$

où yest le point générique de la droite xx'.

Donc la droite en question appartient tout'entière à la quadrique f_p.

Il en résulte que toutes les quadriques qui passent par x contiennent la droite xx'.

Supposons maintenant que la Jacobienne du système L_d soit de rang $r-k$ $(k>0)$

Nous pourrons choisir pour coordonnées de S_{r-k} les: $x_0, x_1, \ldots, x_{r-k}$, en annulant toutes les autre coordonnées, c'est à dire en écrivant:

$$x_{r-k+1} = x_{r-k+2} = \cdots = x_r = 0.$$

Les équations des quadriques $f_0, f_1, \ldots, f_r$ seront du type:

$$f_i(x_0, x_1, \ldots, x_{r-k}, 0, 0, \ldots, 0) = 0.$$

Les dérivées partielles:

$$\frac{\partial f_i}{\partial x_s} \quad (i = 0, 1, \ldots, d)$$

pour:

$$x_{r-k+1} = x_{r-k+2} = \cdots = x_r = 0$$

seront toutes nulles. La matrice Jacobienne du système L'_d:

$$\left\| \frac{\partial f_i}{\partial x_s} \right\| \quad \begin{pmatrix} i = 0, 1, \ldots, d \\ s = 0, 1, \ldots, r-k \end{pmatrix}$$

sera identiquément nulle.

Elle ne pourra pas avoir de rang supériéur à $r-k$, parce que ses lignes ne sont qu'en nombre $r-k+1$, elle ne pourra pas avoir de rang inférieur à $r-k$, sinon le point x aurait pour conjugué un S_g avec $g > 0$ et non le seul point x'.

In en résulte que le système L'_d de S_{r-k} a le rang $r-k$.

Cela porte à conclure pour la première partie du théorème que les quadriques de L'_d, qui passent par x, auront en commun la droite xx'.

Puisque nous pouvons dire la même chose pour tous les S_{r-k} qui passent par x, on en déduit que les quadriques de L_d, qui passent par x, auront en commun S_{k+1} joignant le point x avec S_k.

Toujours dans l'hypothèse que le système soit conjoint de première espèce, considérons les cas $r < d$, en soulignant la condition $r-k \leqslant d$, qui jusqu'ici était superflue.

Il est évident que dans ce cas il doit être $k \geqslant 1$.

Supposons avant tout: $d = r-1$. Considérons le système $L_{r-1/r-k}$ avec $k \geqslant 1$. Par un point générique P de S_r formons un hyperplan, qui coupe $L_{r-1/r-k}$ suivant un système L' de quadriques de S_{r-1}, qui a la même dimension $r-1$ et le même rang: $r-k = (r-1)-(k-1)$ (*v.*: [7]). Puisque la dimension du système est égale à celle de l'hyperplan, selon la première partie du théorème, les quadriques de L'_{r-2}, qui passent par P,-ont en commun un S_k, non multiple. En variant l'hyperplan par P on obtient un'infinité de S_k qui constituent une variété commune à toutes les quadriques de L_{r-1} donné, qui passent par P.

Cette variété ne neut pas être de dimension supérieure à $k+1$ et d'ordre plus grand que 1, autrément dit son intersection avec un hyperplan par P ne serait pas un seul S_k, non plus multiple. Donc cette variété est un S_{k+1}.

Supposons qu'on ait un système $L_{r-2/r-k}$ avec $k \geqslant 2$. Par un point générique P de

S_{r-2}, qui coupe le précédent système suivant un $L''_{r-2/r-k}$. D'après la première partie du théorème, nous pouvons déduire que les quadriques de L'', qui passent par P, ont en commun un S_{k-1}. En variant S_{r-2} et l' S_{r-1} on obtient un S_{k+1} commun à toutes les quadriques de L_{r-3}.

En continuant avec un raisonnement analogue, nous réussirons à démontrer le théorème pour un $L_{r-h/r-k}$ c'est à dire pour un $L_{d/r-k}$ avec: $d \leqslant r-1$, $r-k \leqslant d$. En effet il suffit de supposer $r-d=h$ et évidemment $L_{d/r-k} = L_{r-h/r-k}$.

II) Supposons maintenant que le système L_d soit conjoint de seconde espèce ($r \lneqq d$).

Examinons le cas particulier où les systèmes subordonnés:

$$L_{d_1/m_1}, L_{d_2/m_2}, \ldots, L_{d_s/m_s} \quad (m_i \leqslant d_i)$$

contenus dans L_d et constituants une chaine, sont tous conjoints de première espèce.

Puisque le rang de la Jacobienne est $r-k \leqslant d$ ($k \geqslant 0$), parmi les $d+1$ quadriques linéairement indépendantes, qui individualisent L_d, il y en a $r-k$ fonctionellement indépendantes.

Puisque L_{d_1/m_1} est un système conjoint de première espèce, il satisfait à la première partie de théorème. Pour cela les quadriques de L_{d_1}, qui passent par un générique point P de S_r, ont en commun un S_{r-m_1+1}. Elles constituent un système L_{d_1-1}.

Considérons le système L_{d-1} de quadriques de L_d, qui passent par P.

Les quadriques fonctionellement indépendantes de ce système, qui n'appartiennent pas à L_{d_1-1} sont: $r-k-m_1$. En effet soient:

$$\begin{aligned} &\rho_0 f_0 + \rho_1 f_1 + \cdots + \rho_{d_1} f_{d_1} = 0 \\ &\sigma_0 f_0 + \sigma_1 f_1 + \cdots + \sigma_{d_1} f_{d_1} + \sigma_{d_1+1} f_{d_1+1} + \cdots + \sigma_d f_d = 0 \end{aligned} \tag{17}$$

les équations de L_{d_1} et de L_d.

Si nous imposons à les quadriques des deux systèmes de passer par P, en récherchant ρ_0 et σ_0, on obtient:

$$\begin{aligned} &\rho_1\left(f_1 - \frac{f_1(P)}{f_0(P)} f_0\right) + \rho_2\left(f_0 - \frac{f_2(P)}{f_0(P)} f_0\right) + \cdots \\ &+ \rho_{d_1}\left(f_{d_1} - \frac{f_{d_1}(P)}{f_0(P)} f_0\right) = 0 \\ &\sigma_1\left(f_1 - \frac{f_1(P)}{f_0(P)} f_0\right) + \sigma_2\left(f_0 - \frac{f_2(P)}{f_0(P)} f_0\right) + \cdots + \sigma_{d_1}\left(f_{d_1} - \frac{f_{d_1}(P)}{f_0(P)} f_0\right) \\ &+ \sigma_{d_1+1}\left(f_{d_1+1} - \frac{f_{d_1+1}(P)}{f_0(P)} f_0\right) + \cdots + \sigma_d\left(f_d - \frac{f_d(P)}{f_0(P)} f_0\right) = 0. \end{aligned} \tag{18}$$

Ici le nombre des quadriques fonctionellement indépendantes de L_{d-1} qui

Ils entrecoupent S_{r-m_1+1} suivant un S_{k+1}, qui résulte au moins tangent à toutes les quadriques de L_{d-1}.

Ce résultat est donc indépendant par rapport à d_1 et m_1.

Pour cela en raisonnant analoguément sur $L_{d_2/m_2}, L_{d_3/m_3}, \ldots, L_{d_s/m_s}$ on obterra les espaces:

$$S_{r-m_2+1}, S_{r-m_3+1}, \ldots, S_{r-m_s+1} \tag{19}$$

qui entrecoupés par les hyperplans polaires des restantes quadriques fonctionellement indépendantes de L_{d-1}, donnent toujours le même S_{k+1}. S_{k+1} résulte commun respectivement à toutes les quadriques de $L_{d_1-1}, L_{d_2-1}, \ldots, L_{d_s-1}$ parce qu'il est conténu dans les:

$$S_{r-m_1+1}, S_{r-m_2+1}, \ldots, S_{r-m_s+1}$$

précédents.

Mais puisque $L_{d_1}, L_{d_2}, \ldots, L_{d_s}$ forment dans L_d une chaine, pour le LEMME il est toujours possible choisir dans: $L_{d_1-1}, L_{d_2-1}, \ldots, L_{d_s-1}$, d quadriques linéairement indépendantes, qui individualisent L_{d-1}.

Ces d quadriques possèdent en commun l' S_{k+1} et pour cela toutes les quadriques de L_{d-1} auront en commun S_{k+1}, comme il fallait démontrer.

Le raisonnement précedent a été fait dans l'hypothèse que le système linéaire conjoint de seconde espèce possède seulement des systèmes linéaires subordonnés conjoints de première espèce. Maintenant nous pouvons supposer que les systèmes subordonnés soient tous ou en partie de seconde espèce, en possedant ces derniers des systèmes conjoints de première espèce.

Pour la démonstration faite rien change dans le raisonnement précédent et pour cela la démonstration est la même aussi dans ce cas. Ainsi continuant il est évident que rien change dans le raisonnement précédent quand même les systèmes subordonnés des systèmes subordonnés sont d'un'espèce quelconque et la démonstration est valable pour tous les systèmes conjoints de seconde espèce avec $r \geqq d$.

De cette manière le théorème résulte démontré complètement.

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Aligned magnetic effects through varying permeable bed

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Abstract. The aligned magnetic effects on a steady laminar, viscous, incompressible, conducting fluid down an open inclined channel bounded below by a bed of varying permeability has been studied when the free surface is exposed to atmospheric pressure. Beavers and Joseph slip condition at the interface of the free flow region and the fluid flow in the porous bed and the Darcy's law in the porous medium have been used. The expressions for velocity, magnetic strength and the mass flow across the cross-section of the channel are obtained.

Keywords. Aligned magnetic effect; varying permeability.

Nomenclature

$F = U_0^2/gh$	Froude number
$-F(\mathrm{d}p/\mathrm{d}x)$	pressure gradient
g	acceleration due to gravity
H_x	magnetic field
K_0	permeability of the medium
$M^2 = (\sigma_e \mu_e^2 H_0^2 h^2)/\mu$	Hartmann number
p	pressure
Q	velocity in the porous medium
$R = U_0 h/\nu$	Reynold's number
$R_m = U_0 h/\eta_e$	magnetic Reynold's number
$(u, 0, 0)$	velocity components at a point
α	slip parameter
η_e	magnetic diffusivity
μ	coefficient of viscosity
μ_e	magnetic permeability
ν	kinematic coefficient of viscosity
ρ	fluid density
σ_e	conductivity of the medium

1. Introduction

The flow of viscous fluid in an inclined channel with free surface has applications in coating in paper rolls, hydraulic engineering, agricultural engineering and in designs

of the drainage and irrigation canals. Satyaprakash [2] examined analytically the viscous flow down an open inclined channel with impermeable bed. Verma and Vyas [5] studied viscous flow down an open inclined channel with naturally permeable bed. Venkataramana and Bathaiah [4] studied the hydromagnetic viscous flow down an open inclined channel with naturally permeable bed, without taking the effect of magnetic field in the porous bed. Chandna *et al* [1] obtained some exact solutions in a steady, viscous, incompressible and variably inclined MHD plane flow. Sinha and Chadda [3] examined the steady laminar viscous flow down an open inclined channel with a porous bed of varying permeability. In this paper, the effect of aligned magnetic field in a laminar flow of viscous, incompressible conducting fluid down an open inclined channel with a porous bed of varying permeability is studied. The velocity and the flux across the cross-section of the channel are obtained.

2. Formulation of the problem

A steady laminar flow of viscous conducting fluid down an open inclined channel of depth h and width b under the influence of an aligned magnetic field is considered. The walls of the channel are normal to the surface of bed and the bed is taken to be inclined at an angle β $(0 \leqslant \beta \leqslant \pi/2)$ to the horizontal and is of varying permeability. The permeability decays exponentially as the depth of the bed increases. It is taken as

$$K_0 = K \exp(-\sigma y). \tag{1}$$

The flow is due to constant pressure gradient at the mouth of the channel. The velocity components in the Cartesian coordinate system are $(u, 0, 0)$. A magnetic field of strength H_0 is applied at an angle θ to the porous bed (x-axis). The length of the porous bed is assumed to be long enough to make $\partial/\partial x$ of all physical quantities, except pressure, zero. The governing equation in the free field region is

$$\nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g \sin\beta - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu_e}{\rho} H_y \frac{\partial H_x}{\partial y} = 0. \tag{2}$$

The pressure can be found from the equations in y- and z-directions after knowing H_x, the x-component of the induced magnetic field. The field equation is

$$H_y \frac{\partial u}{\partial y} + \frac{1}{\eta_e}\left(\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2}\right) = 0. \tag{3}$$

The y-component H_y of the induced magnetic field becomes constant, i.e.

$$H_y = -H_0 \sin\theta. \tag{4}$$

The Darcy's equation in the porous region is

$$K_0\left(\frac{\partial p}{\partial}\right.$$

reduce to

$$K_0 = K \exp(-\sigma y) \tag{6}$$

$$\Delta^2 u = A + (M^2/R_m) \sin\theta (\partial H_x/\partial y) \tag{7}$$

$$\Delta^2 H_x = R_m \sin\theta (\partial u/\partial y) \tag{8}$$

and

$$Q = \frac{-KA \exp(-\sigma y)}{1 + KM^2 \exp(-\sigma y) \sin^2\theta} \tag{9}$$

where $\Delta = (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$ and

$$A = \left(R \frac{\partial p}{\partial x} - \frac{R}{F} \sin\beta \right).$$

The boundary conditions are

$$u = 0, \quad H_x = 0 \quad \text{at } z = \pm\gamma (\gamma = b/h), \tag{10}$$

$$\partial u/\partial y = 0 \quad \text{at } y = 0, \tag{11}$$

$$\partial u/\partial y = -\frac{\alpha}{\sqrt{K}} \exp(\sigma y/2)(u - Q) \quad \text{at } y = 1, \tag{12}$$

$$\begin{aligned} &H_x = -\cos\theta \quad \text{at } y = 1, \\ &H_x = 0 \quad \text{at } y = 0. \end{aligned} \tag{13}$$

3. Solution of the problem

Substituting $z = (2\gamma\xi/\pi) - \gamma$ in (7) and (8), the equations reduce to

$$\frac{\partial^2 u}{\partial y^2} + \frac{\pi^2}{4\gamma^2} \frac{\partial^2 u}{\partial \xi^2} = A + \frac{M^2}{R_m} \frac{\partial H_x}{\partial y} \sin\theta, \tag{14}$$

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\pi^2}{4\gamma^2} \frac{\partial^2 H_x}{\partial \xi^2} = R_m \sin\theta \frac{\partial u}{\partial y}, \tag{15}$$

with the boundary conditions (10) becoming

$$u = 0, \quad H_x = 0 \quad \text{at } \xi = 0 \quad \text{and} \quad \xi = \pi. \tag{16}$$

Let finite sine transform of a function f be defined as

$$\tilde{f} = \int_0^\pi f \sin n\xi \, d\xi. \tag{17}$$

$$(d^2\tilde{u}/dy^2) - \lambda^2\tilde{u} = B + \frac{M^2}{R_m}\sin\theta(d\tilde{H}_x/dy). \tag{18}$$

$$(d^2\tilde{H}_x/dy^2) - \lambda^2\tilde{H}_x = R_m \sin\theta(d\tilde{u}/dy) \tag{19}$$

ere $\lambda^2 = (n^2\pi^2/4\gamma^2)$, $B = A(1 - \cos n\pi)/n$,

$$\tilde{Q} = \frac{-KB\exp(-\sigma y)}{(1 + KM^2\exp(-\sigma y)\sin^2\theta)}, \tag{20}$$

$$(d\tilde{u}/dy) = 0 \quad \text{at } y = 0, \tag{21}$$

$$(d\tilde{u}/dy) = -\frac{\alpha}{\sqrt{K}}\exp(\sigma y/2)(\tilde{u} - \tilde{Q}) \quad \text{at } y = 1, \tag{22}$$

$$\tilde{H}_x = -\cos\theta(1 - \cos n\pi)/n \quad \text{at } y = 1, \tag{23}$$

$$\tilde{H}_x = 0 \quad \text{at } y = 0. \tag{24}$$

ıe following equation is obtained on decoupling (18) and (19)

$$(d^4\tilde{u}/dy^4) - (2\lambda^2 + M^2\sin^2\theta)(d^2\tilde{u}/dy^2) + \lambda^4\tilde{u} = -B\lambda^2. \tag{25}$$

e solution of (25) is

$$\tilde{u} = A_1\cosh p_1 y + A_2\cosh p_2 y - B/\lambda^2, \tag{26}$$

ere $p_{1,2}$ are the positive real roots of the auxiliary equation of (25),

$$A_1 = (A_n M_2\sinh p_2 - E_n C_n)/B_n,$$

$$A_2 = (D_n C_n - A_n M_1\sinh p_1)/B_n,$$

$$A_n = \frac{B\alpha\exp(\sigma/2)}{\sqrt{K}}\left(\frac{K\exp(-\sigma)}{1 + KM^2\exp(-\sigma)\sin^2\theta} - \frac{1}{\lambda^2}\right),$$

$$C_n = \frac{M^2}{2R_m}\left(\frac{1 - \cos n\pi}{n}\right)\sin 2\theta,$$

$$D_n = p_1\sinh p_1 + \frac{\alpha}{\sqrt{K}}\exp(\sigma/2)\cosh p_1,$$

$$E_n = p_2\sinh p_2 + \frac{\alpha}{\sqrt{K}}\exp(\sigma/2)\cosh p_2,$$

$$B_n = E_n M_1\sinh p_1 - D_n M_2\sinh p_2,$$

$$M_1 = (p_1^2 - \lambda^2)/p_1 \quad \text{and} \quad M_2 = (p_2^2 - \lambda^2)/p_2.$$

H_x is obtained by using (26) in (18) as

$$\tilde{H}_x = \frac{R_m}{M^2 \sin\theta}[M_1 A_1 \sinh p_1 y + A_2 M_2 \sinh p_2 y]. \tag{27}$$

Using the inversion formula for finite transform, (26) and (27) become

$$u = \frac{2}{\pi}\sum_{n=1}^{\infty}(A_1 \cosh p_1 y + A_2 \cosh p_2 y - B/\lambda^2)\sin n\xi \tag{28}$$

$$H_x = \frac{2}{\pi}\frac{R_m}{M^2 \sin\theta}\sum_{n=1}^{\infty}(M_1 A_1 \sinh p_1 y + M_2 A_2 \sinh p_2 y)\sin n\xi. \tag{29}$$

If in (28), $K \to 0$ i.e. the bed is impermeable, the velocity profiles are given by

$$u = \frac{2}{\pi}\sum_{n=1}^{\infty}(A_3 \cosh p_1 y + A_4 \cosh p_2 y - B/\lambda^2)\sinh n\xi \tag{30}$$

where

$$A_3 = -[(B/\lambda^2)M_2 \sinh p_2 + C_n \cosh p_2]/A_5,$$

$$A_4 = [(B/\lambda^2)M_1 \sinh p_1 + C_n \cosh p_1]/A_5,$$

$$A_5 = M_1 \sinh p_1 \cosh p_2 - M_2 \sinh p_2 \cosh p_1.$$

The flux across the cross-section of the channel perpendicular to the x-axis is given by

$$v = \int_{z=-\gamma}^{\gamma}\int_{y=0}^{1} u\,dy\,dz$$

$$= \frac{4\gamma}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n}\right)\left[\frac{A_1}{p_1}\sinh p_1 + \frac{A_2}{p_2}\sinh p_2 - \frac{B}{\lambda^2}\right]. \tag{31}$$

In the case of impermeable bed, the flux is given by

$$v = \frac{4\gamma}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n}\right)\left[\frac{A_3}{p_1}\sinh p_1 + \frac{A_4}{p_2}\sinh p_2 - B/\lambda^2\right]. \tag{32}$$

4. Discussion

The figures show the behaviour of the flow field for different values of pressure gradient and Hartmann number under different permeability conditions associated with the bed of the channel. The graphs are drawn for $\beta = \pi/6$, $\gamma = \pi/2$, $k = 0$ and 10^{-4}, $\alpha = 0{\cdot}01$, $\sigma = 0$ to $0{\cdot}3$, $R = F = R_m = 1$. Figure 1 illustrates the velocity profiles at the central cross-section with different values of pressure gradient and Hartmann number for fixed θ (in radians). When the pressure gradient and Hartmann number increases the velocity increases for the non-zero permeability. But in the case of impermeable bed the velocity increases for different values of pressure gradient and

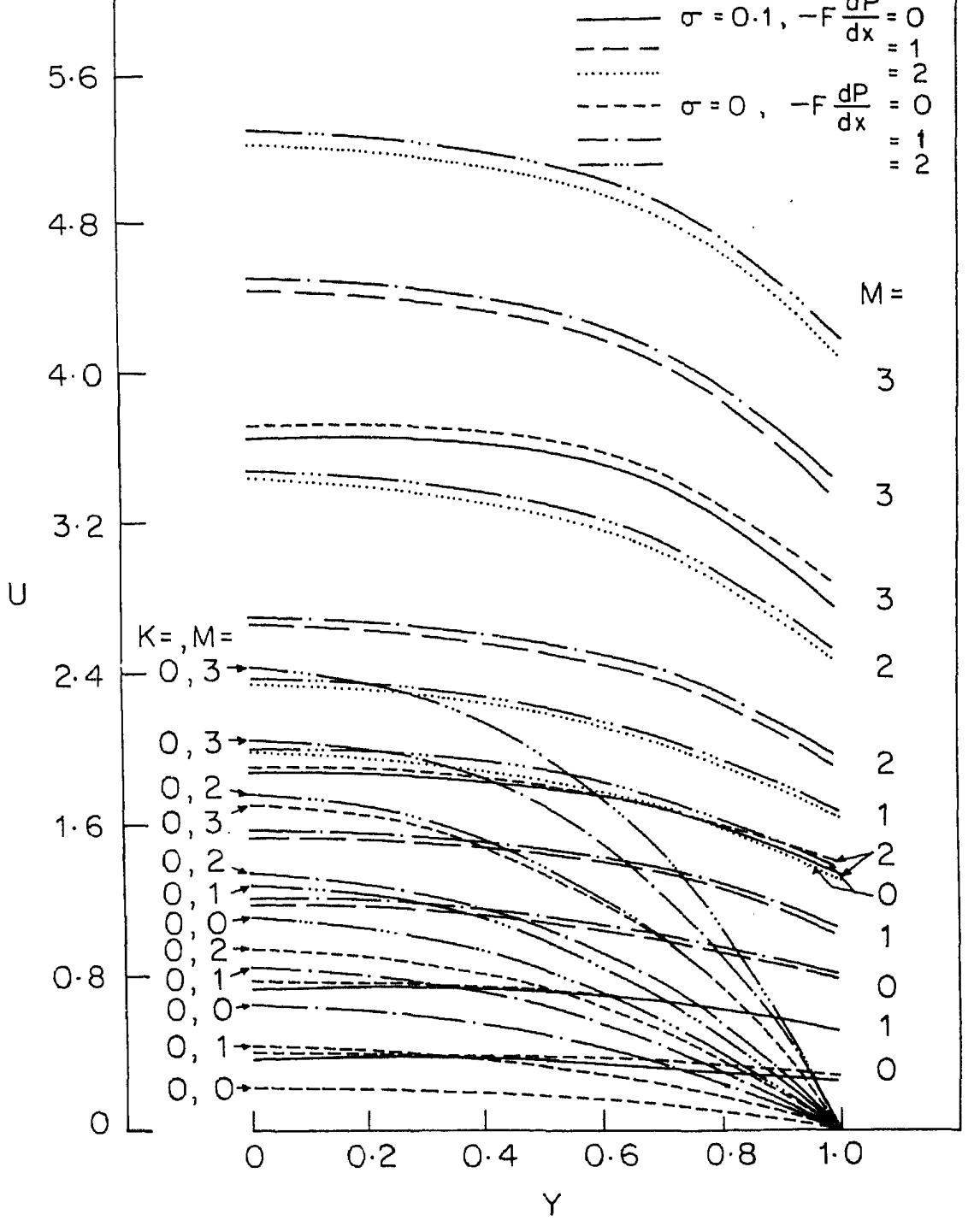

Figure 1. Velocity profiles for $\theta = \pi/4$, $K = 10^{-4}$.

Figure 2 shows the velocity profiles for different values of θ for fixed pressure gradient and Hartmann number. In this figure it is clear that the velocity is maximum at $\theta = 0{\cdot}80$ and becomes zero at $\theta = 2{\cdot}014$ for both permeable and impermeable bed. As the inclination (θ) decreases from $\theta = \pi/2$, the flow velocity increases and attains maximum velocity at $\theta = 0{\cdot}80$. When the magnetic field is applied in the aligned direction ($\theta < \pi/2$) the flow velocity is larger than the velocity when the magnetic field is applied in the transverse direction. When $\theta > \pi/2$, the flow velocity is very less in the aligned magnetic field than the value obtained in the transverse magnetic field. The velocity becomes zero as θ increases. To remove the back flow the aligned magnetic field should be applied such that the angle of inclination of the magnetic

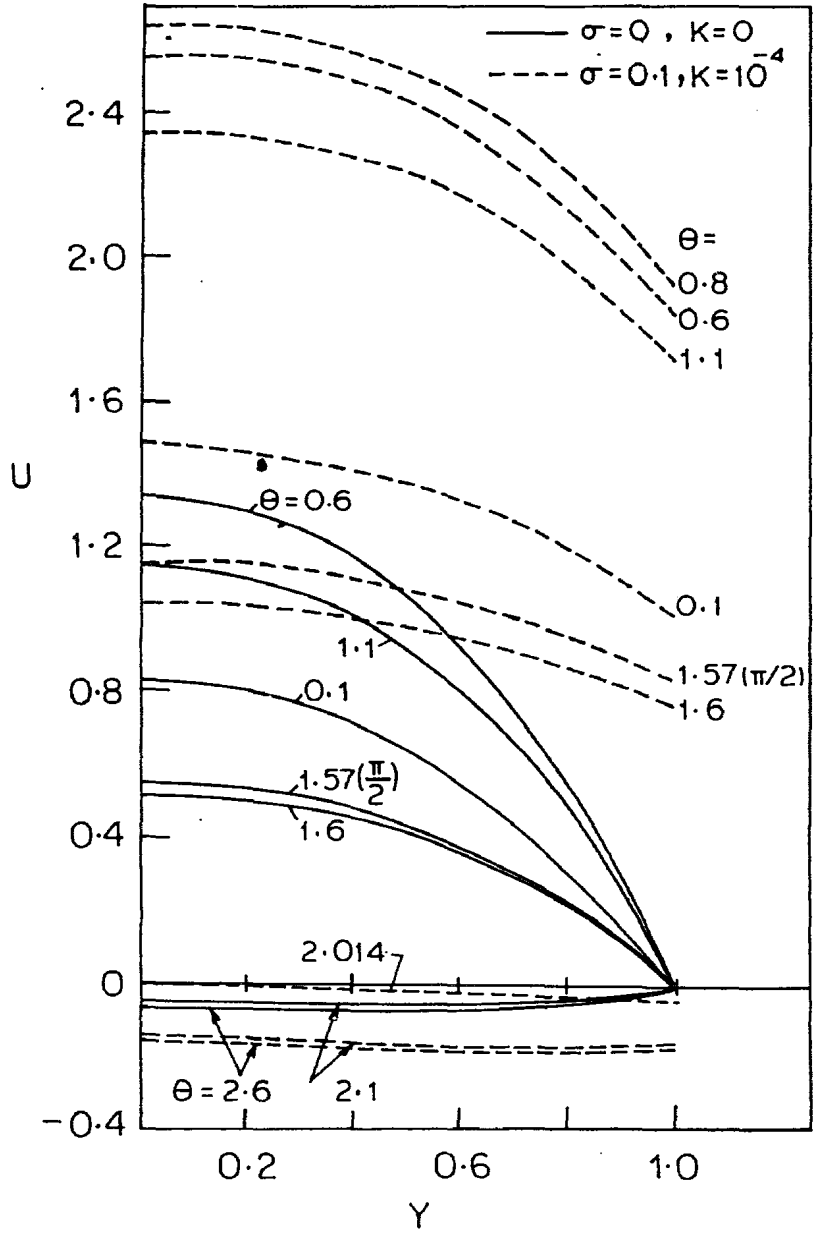

Figure 2. Velocity profiles for $M = 2$, $-F(\mathrm{d}p/\mathrm{d}x) = 1$.

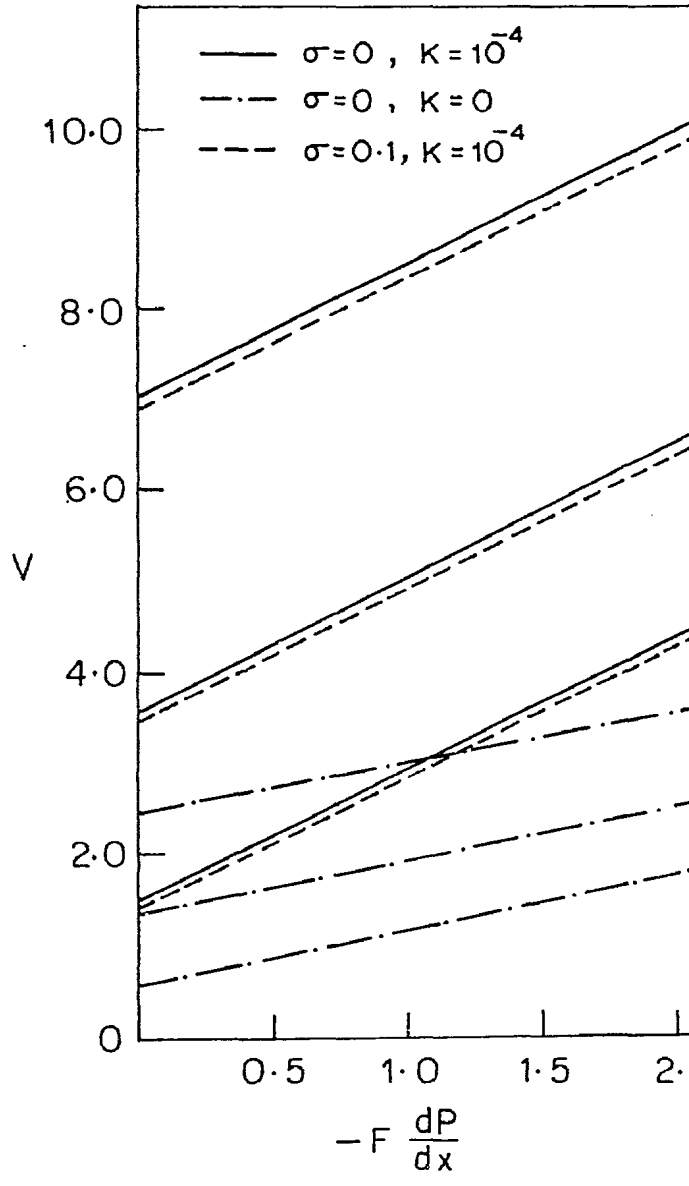

Figure 3. Effect of magnetic field for $M = 2$, $-F(\mathrm{d}p/\mathrm{dx})$

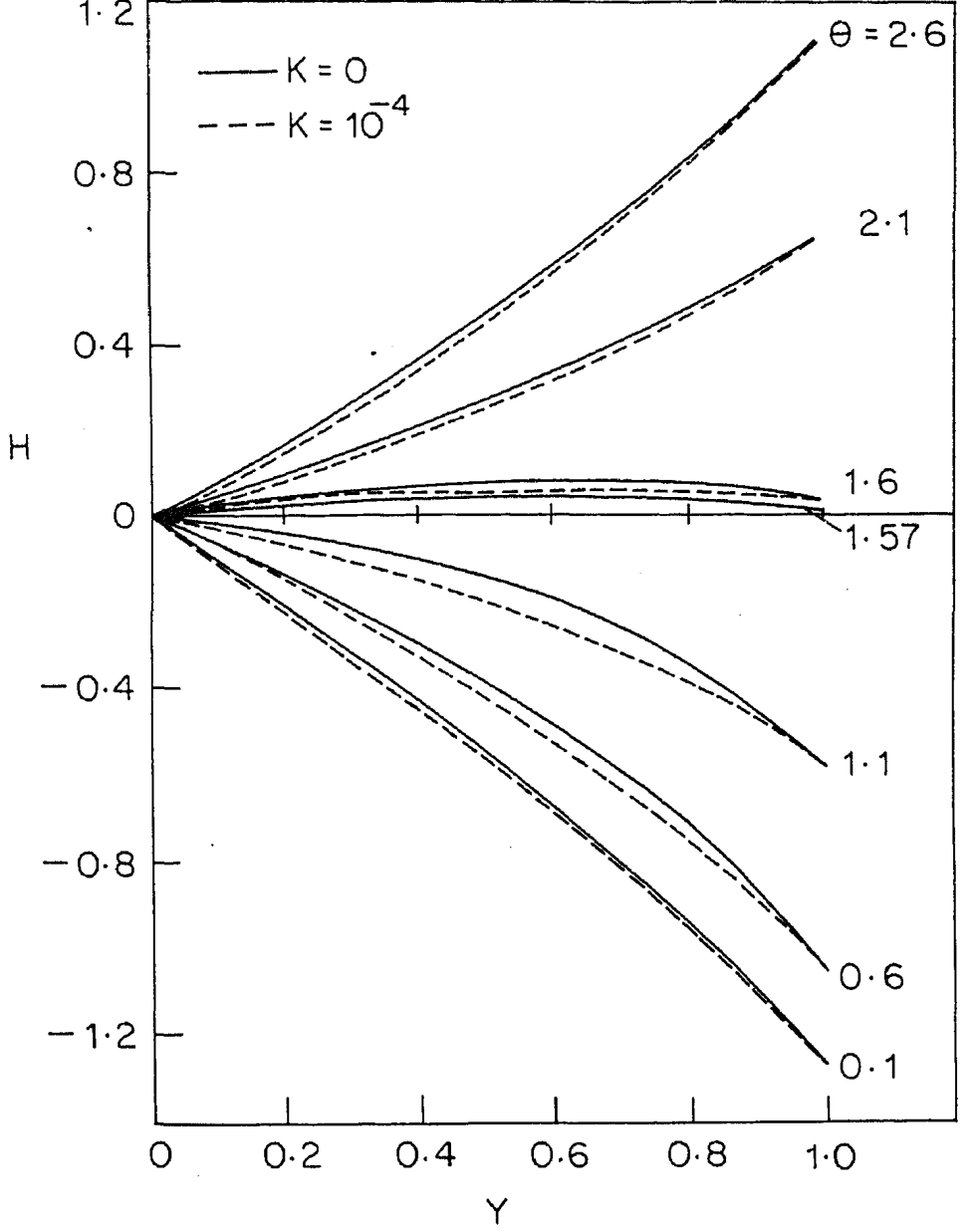

Figure 4. Flux across the transverse cross-section of the channel.

Figure 4 shows the flux across the cross section perpendicular to the length of the channel for different values of Hartmann number. It is seen that the flux is greater when the bed is permeable than when it is impermeable, but for exponentially varying permeability this deviation is attenuated.

In figure 3 the magnetic strength H_x is described for different inclinations θ of the magnetic lines to the bed of the inclined plane, when the Hartmann number is fixed as $M = 2$. When the applied magnetic field is applied in the direction $0 < \theta < \pi/2$, the magnetic strength H_x decreases from negative value to zero. When the magnetic field is in transverse direction, the magnetic strength is positive and it goes on increasing as the angle of inclination increases from $\pi/2$ to π.

In figure 5, the velocity profiles for different values of porosity parameter σ are drawn. In general, it is observed that the porosity σ decreases the velocity of the flow. On comparison, it is observed that the porosity dominates over the magnetic

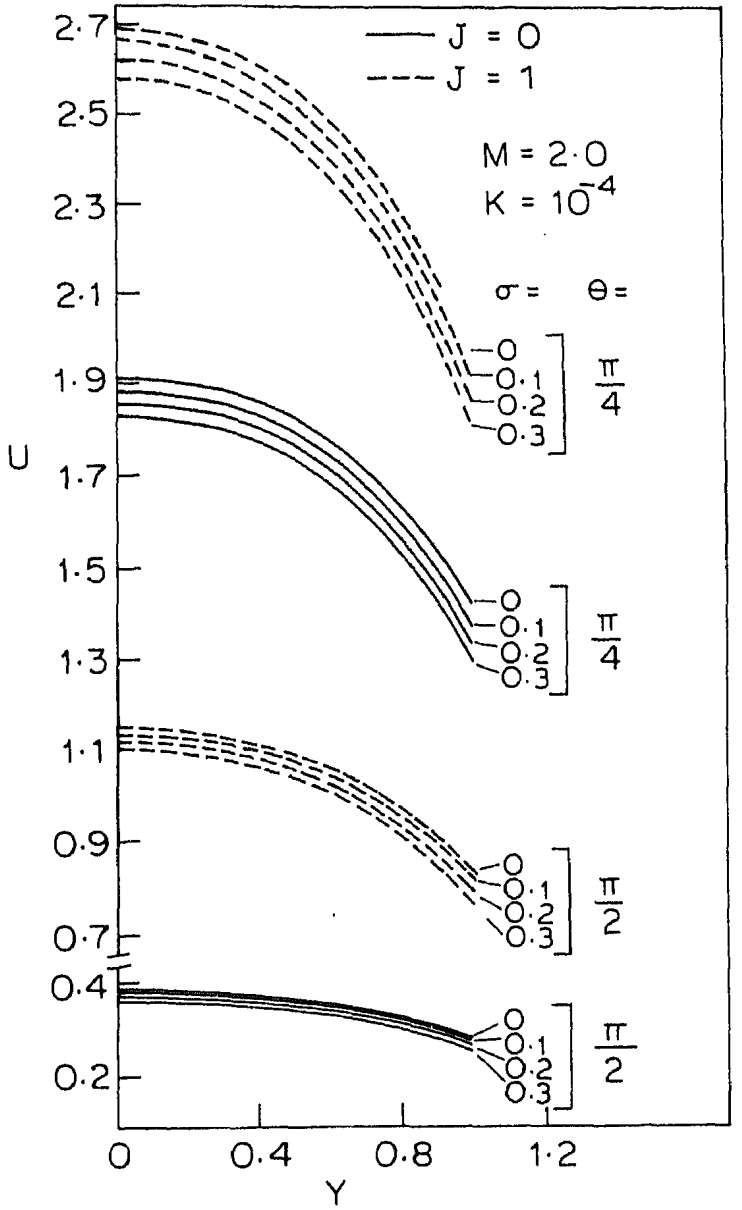

Figure 5. Velocity profiles for different values of porosity parameter σ.

ratio of the velocity for aligned case ($\theta = \pi/4$) to the velocity for the transverse case ($\theta = \pi/2$) decreases as the pressure gradient (J) increases.

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On shock dynamics

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Abstract. In a recent paper, Srinivasan and Prasad criticize certain approximations that were made in early work on shock dynamics. The criticism however is based on certain misinterpretations. This short note offers the correct interpretations and explains the issues.

Keyword. Shock dynamics.

The basic equations of the theory of shock dynamics developed earlier (Whitham [3]) may be taken in the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}\eta}=\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}s}, \tag{1}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}=-\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}\eta}, \tag{2}$$

where M is the shock Mach number, A the ray tube area, θ the local ray direction and $\mathrm{d}/\mathrm{d}s$ and $\mathrm{d}/\mathrm{d}\eta$ refer to the derivatives along and transverse to the rays. (In the original work, a true co-ordinate system (α, β) was preferred, but since the line elements are $\mathrm{d}s = M\,\mathrm{d}\alpha, \mathrm{d}\eta = A\mathrm{d}\beta$, the two forms are equivalent. One has to remember that $\mathrm{d}/\mathrm{d}s$ and $\mathrm{d}/\mathrm{d}\eta$ do not commute).

These equations are exact and follow immediately from the geometry of the successive shock positions and their orthogonal system of rays.

Srinivasan and Prasad [2] note that, with some extension, relations obtained by Maslov [1] can be put in the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}\eta}=\frac{1}{B_1}\frac{\mathrm{d}\mu}{\mathrm{d}s}-\frac{B_2}{B_1}\frac{\mathrm{d}\mu}{\mathrm{d}N}, \tag{3}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}=-\frac{1}{D}\frac{\mathrm{d}\mu}{\mathrm{d}\eta}. \tag{4}$$

Here $\mu=(\rho-\rho_0)/\rho_0$ is the density ratio at the shock and B_1, B_2 and D are the functions of μ. The derivative $\mathrm{d}\mu/\mathrm{d}N$ in (3) refers to the instantaneous value of the space derivative

$$\frac{d\mu}{ds} = \frac{d\mu}{dN} + \frac{1}{U}\frac{\partial \mu}{\partial t}, \tag{5}$$

where $\partial\mu/\partial t$ is the local time derivative and U is the normal shock velocity.

The Mach number M and the density μ are related by the shock condition

$$M^2 = \frac{1+\mu}{1-\frac{\gamma-1}{2}\mu} \tag{6}$$

and it is readily checked that (2) and (4) are the same.

It is less immediate but still relatively straightforward to show directly from the equations of motion that (1) and (3) agree. In fact, this seems to be a much quicker way to derive (3) instead of the long calculations required in [2].

Both sets of equations (1)–(2) and (3)–(4) are open systems which require further relations to complete them. In [3], based strongly on the geometrical setting and analogy with geometrical optics, a functional relation between A and M was added to (1)–(2) to provide an approximate closure. From (6), this could be recast as a relation between A and μ. If this is used in (1), we have

$$\frac{d\theta}{d\eta} = \frac{1}{b(\mu)}\frac{d\mu}{ds}, \tag{7}$$

where $b(\mu)$ is a known function of μ.

The correct interpretation of this approximation, when viewed from (3), is that $d\mu/dN$ and $d\mu/ds$ are assumed to be proportional to each other. That is

$$\frac{d\mu}{dN} = a(\mu)\frac{d\mu}{ds}. \tag{8}$$

This would, in any case, be a natural way to close the system (3)–(4), and would appear to be the only simple one. The two closures would be identical if

$$a = \left(1 - \frac{B_1}{b}\right)\frac{1}{B_2}. \tag{9}$$

Thus, the earlier shock dynamic theory not only provides a natural equivalent closure on Maslov's equations, but also provides a formula for the coefficient $a(\mu)$ in (8).

Srinivasan and Prasad are wrong on two counts. They assume that the second term with coefficient B_2 in (3) is neglected and that the first coefficient B_1 is taken incorrectly. In fact the two terms are combined as in (7) with a revised coefficient

$$b = B_1/(1 - aB_2). \tag{10}$$

They then make much of the difference between b and B_1 in their criticism of the

success; it shows that the theory retains and determines the effect of the term $d\mu/dN$. The agreement of B_1 and b would have been the disaster!

There is one amusing point in their tabulation in table 1 of the coefficients $B_1(\mu)$ and $b(\mu)$, equal to their $B_3(\mu)$. The final entry for $\mu = 5$ is singled out presumably to show the most dramatic instance of the "error"; the coefficients are given as

$$B_1 = 0{\cdot}12417634 \times 10^{-7},$$

$$B_3 = 0{\cdot}74066669 \times 10^{-7}.$$

With perhaps some satisfaction, the authors point out the six-fold discrepancy. A quick glance at the formulae shows that both values are in fact zero. The discrepancy is therefore hardly surprising.

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Nonlinear random equations involving completely closed operators

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Abstract. We introduce the concept of completely closed operators in Banach spaces and then obtain the existence of random solutions of operator equations involving such operators. As simple corollaries we obtain the existence theorems for random operator equations involving monotone operators as well as operators of type (M).

Keywords. Nonlinear random equations; closed operators.

1. Introduction

The study of random operator equations was first initiated by Spacek [21] and Hans [9] with a systematic development of the theory of random fixed points. Subsequently these results were extended and applied to various problems by Hans [10, 11] and Bharucha-Reid [2, 3]. Now there is a wide range of literature in this field, for reference see Bharucha-Reid [4, 5].

In this paper we first introduce the concept of completely closed operators in Banach spaces and then obtain random solvability results for operator equations involving such operators. In studying such a class of operators we cover some important operators such as maximal monotone operators, operators of type (M), completely continuous operators and compact perturbations of the identity operator. As simple corollaries we obtain the measurability inverse theorems of Hans [10], Joshi [13], Nashed and Salehi [18] and random analogs of fixed point theorems of Banach and Schauder.

In § 2 we introduce the necessary basic concepts of the theory of nonlinear functional analysis and theory of random operators. Section 3 deals with existence results. We conclude the paper in §4 by giving the application of our theory to random Hammerstein operator equation including a concrete example.

2. Preliminaries

Throughout this paper (Ω, β, μ) denotes a complete probability measure space and X a separable real Banach space with dual X^*, $(.,.)$ denoting the duality pairing between X and X^*. If X is a Hilbert space then $(.,.)$ indicates the inner product in X. For any subset D of X, 2^D denotes the family of all subsets of D.

A mapping $T:\Omega \to 2^X$ is said to be measurable, if for any closed subset C of X, $T^{-1}(C) = \{\omega \in \Omega: T(\omega) \cap C \neq \phi\}$ is in β. A measurable mapping $g:\Omega \to X$ is said to be a

valued measurable mapping is called a random variable. $T:\Omega \times D \to X$ is called a random operator if for any $x \in D$, $T(\omega)x$ is a random variable. If $T:\Omega \times D \to X$ is a random operator which is 1–1 and onto for each $\omega \in \Omega$ then $T^{-1}:\Omega \times X \to D$ is defined and is said to be random if $T^{-1}(\omega)y$ is a random variable for any $y \in X$. A random operator $T:\Omega \times D \to X$ is called separable if there exists a countable dense set S in D and $N \in \beta$ with $\mu(N) = 0$ such that for every closed subset K of X and open subset O of D we have

$$\{\omega \in \Omega: T(\omega)(O \cap S) \subset K\} \Delta \{\omega \in \Omega: T(\omega)O \subset K\} \subset N.$$

Remark. Separability of the operator $T:\Omega \times D \to X$ does not imply the joint measurability of T as a function of (ω, x) with respect to the product σ-field on $\Omega \times D$ as we see by the following example.

Example. Let $\{X_t, t \geqslant 0\}$ be a stochastic process on (Ω, β, μ) such that for each t, X_t has the Gaussian distribution with 0 mean and unit variance and $EX_tX_s = 0$, if $s \neq t$. One can look into X_t as a random operator $T(\omega)$ from $D = \{t \geqslant 0, t \in R\} \subset R$ to R. Example 3.8.5 (page 534) of Billingsley [5a] shows that X_t is a separable process and hence $T(\omega)$ is a separable operator. However example 1.2.5 of Kalianpur [14a] shows that this process is not measurable, i.e. T is not a measurable mapping.

An X-valued random variable $x(\omega)$ is said to be a random solution of the random operator equation $T(\omega)x(\omega) = y(\omega)$, where $y(\omega)$ is a given random variable, if it satisfies the relation $\mu\{(\omega: T(\omega)x(\omega) = y(\omega)\} = 1$. A random operator $T:\Omega \times D \to X$ is said to be a random contraction if there exists a random constant $k(\omega) < 1$ such that $\| T(\omega)x - T(\omega)y \| \leqslant k(\omega) \| x - y \|$ for all $x, y \in X$.

A mapping $T:X \to Y$ (Y is another real Banach space) is said to be bounded if it maps bounded subsets of X into bounded subsets of Y. T is called compact if it maps bounded sets into precompact sets. T is demicontinuous if $x_n \to x$ in X implies that Tx_n converges weakly to Tx (denoted by $Tx_n \rightharpoonup Tx$) in Y. T is hemicontinuous if $x_n \to x$ along the lines in X implies that $Tx_n \rightharpoonup Tx$. T is weakly continuous if $x_n \rightharpoonup x$ implies that $Tx_n \rightharpoonup Tx$, T is completely continuous if $x_n \rightharpoonup x$ implies that $Tx_n \to Tx$.

$T:X \to X^*$ is called monotone if $(Tx - Ty, x - y) \geqslant 0$ for all $x, y \in D(T)$. T is called maximal monotone if it has no proper monotone extensions, that is, if for $[x, y] \in X \times X^*$ we have $(y - Tu, x - u) \geqslant 0$ for all $u \in D(T)$, then $x \in D(T)$ and $y = Tx$. Every demicontinuous monotone operator with dense domain is maximal monotone, refer [6]. A bounded linear monotone operator $A: X \to X^*$ is said to be angle-bounded with constant $c \geqslant 0$ if

$$|(Ax, y) - (Ay, x)| \leqslant 2c[(Ax, x)(Ay, y)]^{1/2} \quad \text{for all } x, y \in X.$$

It is clear that every symmetric bounded monotone operator is angle-bounded with constant $c = 0$.

$T:X \to X^*$ is said to be of type (M) provided the following two conditions hold:

(i) If $x_n \rightharpoonup x$ in X, $Tx_n \rightharpoonup y$ in X^* and $\limsup_n (Tx_n, x_n) \leqslant (y, x)$, then $Tx = y$.

(ii) T is continuous from finite dimensional subspaces of X into X^* equipped with

and $c(r) \to \infty$ as $r \to \infty$.

A mapping $J: X \to X^*$ is called duality mapping if $(Jx, x) = \|x\| \|Jx\|$ and $\|x\| = \|Jx\|$ for all $x \in X$. The duality mapping is a demicontinuous monotone everywhere defined operator (refer Petryshyn [19]).

A random operator $T: \Omega \times X \to Y$ is called hemicontinuous (demicontinuous, bounded, etc.) if for each $\omega \in \Omega$, $T(\omega)$ is hemicontinuous (demicontinuous, bounded, etc.). Similarly we define random monotone and M-type operator if $T(\omega)$ maps X into X^* for $\omega \in \Omega$.

With these definitions we have

THEOREM 2.1.[16]. Let X and Y be separable Banach spaces and let $T: \Omega \times X \to Y$ be a demicontinuous random operator. Then for any X-valued random variable x, the function $T(\omega, x(\omega))$ is a Y-valued random variable.

THEOREM 2.2.[17]. Let X be a separable complete metric space and let $T: \Omega \to 2^X$ be a closed valued measurable multifunction. Then there exists at least one random selector of T.

THEOREM 2.3.[4]. Let X and Y be separable Banach spaces and $T: \Omega \times X \to Y$ a random operator and S a countable dense subset of X. Then T is separable iff there exists $N \in \beta$ with $\mu(N) = 0$ such that for all $\omega \notin N$ and $x \in X$, there exists $\{x_n\} \subset S$ such that $x_n \to x$ and $T(\omega)x_n \to T(\omega)x$.

THEOREM 2.4.[6]. Let $T: X \to X^*$ be a demicontinuous coercive monotone operator. Then $R(T) = X^*$.

3. Existence theorems

In the following, unless otherwise stated, X denotes a real separable reflexive Banach space with dual X^*, Y another real separable Banach space and $T: X \to Y$ is a operator (not necessarily linear) with domain $D(T)$.

DEFINITION 3.1. $T: D(T) \subset X \to Y$ is said to be completely closed if whenever there exists a sequence $\{x_n\} \subset D(T)$ with $x_n \rightharpoonup x$ in X and $Tx_n \to y$ in Y, then $x \in D(T)$ and $y = Tx$.

Remark 1. It is clear that if T is invertible with T^{-1} demicontinuous then T is completely closed. Also every weakly continuous operator is completely closed and so is negative of a completely closed operator.

Example 3.1. Let $X = L_2[0, 1]$ and $T: X \to X$ be the differential operator $\mathrm{d}/\mathrm{d}t$ with domain $D(T)$ defined as

$$D(T) = \{x \in L_2[0, 1]: \quad x \text{ is absolutely continuous with}$$

T is completely closed.

If T is a linear operator which is everywhere defined on X, then complete closedness of T is equivalent to boundedness.

PROPOSITION 3.1. Let $T:X \to Y$ be a linear operator with $D(T) = X$. Then T is bounded iff T is completely closed.

Proof. Let T be bounded. Since every bounded operator is weakly continuous, it follows that T is completely closed.

Conversely let T be completely closed and let $(x_n, Tx_n) \to (x, y)$ in $X \times Y$. This implies that $x_n \to x$ and $Tx_n \to y$ and hence complete closedness of T gives $y = Tx$. Thus the graph of T is closed and hence by closed graph theorem it follows that T is bounded.

Let T be a map from X into X^*, then monotonicity of T is related to complete closedness in the following way.

PROPOSITION 3.2. Let $T:D(T) \subset X \to X^*$ be a maximal monotone operator. Then T is completely closed.

Proof. Let $\{x_n\}$ be a sequence in $D(T)$ with $x_n \to x$ and $Tx_n \to y$ in X^*. Then monotonicity of T gives

$$(Tx_n - Tx_0, x_n - x_0) \geqslant 0 \quad \text{for all } x_0 \in D(T).$$

Taking limit in the above inequality as $x_n \to x$ and $Tx_n \to y$ we get

$$(y - Tx_0, x - x_0) \geqslant 0 \quad \text{for all } x_0 \in D(T).$$

Now maximal monotonicity implies that $x \in D(T)$ and $y = Tx$.

Remark 2. Every demicontinuous monotone operator with dense domain is completely closed. So the duality mapping $J:X \to X^*$ is a completely closed operator.

The converse of the above proposition is false. For, if T is maximal monotone then $(-T)$ is not maximal monotone, whereas it is completely closed.

Remark 3. It can be easily seen that every operator of type (M) is completely closed but the converse is not true as we will see in example 3.3. Also every completely continuous operator is completely closed but this is not so for compact operators as we see in the following example.

Example 3.2. Let $X = l_2$. Define $T:X \to X$ as

$$Tx = (1 - \|x\|, 0, 0, \ldots).$$

Since T is a finite rank operator, it is compact. But it is not completely closed. For this, consider the sequence $\{x_n\}$ of elements in l_2 as

$$x_1 = (\tfrac{1}{2}, 0, 0, \ldots), \quad x_2 = (0, \tfrac{1}{2}, 0, \ldots), \ldots.$$

Then $x_n \to 0$ and $Tx_n \to (1, 0, 0, \ldots)$ but $T0 \neq (1, 0, 0, \ldots)$.

However, a completely continuous perturbation of a completely closed operator is completely closed.

PROPOSITION 3.3. Let $T:D(T)\subset X\to Y$ be a completely closed operator and $S:X\to Y$ a completely continuous operator with $D(S)=X$. Then $T+S$ is completely closed with domain $D(T)$.

Compact and continuous perturbation of a completely closed operator with continuous inverse is closed.

PROPOSITION 3.4. Let $T:D(T)\subset X\to Y$ be a completely closed operator with T^{-1} continuous and $S:X\to Y$ be a compact and continuous operator with $D(S)=X$. Then $T+S$ is completely closed with domain $D(T)$.

Proof. Let $x_n\in D(S\cap T)=D(T)$ be such that $x_n\to x$ and $Tx_n+Sx_n\to z$. Since $\{x_n\}$ is a bounded sequence and S is compact it follows that $\{Sx_n\}$ has a subsequence $\{Sx_{n_k}\}$ converging to y, which in turn implies that $\{Tx_{n_k}\}$ converges to $z-y$. Since T^{-1} is continuous, it follows that $x_{n_k}\to T^{-1}(z-y)\in D(T)$. Uniqueness of limit implies that $x=T^{-1}(z-y)$ or $Tx=z-y$. So $x_{n_k}\to x$ and hence $Sx_{n_k}\to Sx$ which together with $Tx_{n_k}\to z-y=Tx$ implies that $Tx_{n_k}+Sx_{n_k}\to Tx+Sx=z$. This proves the assertion.

If both X and X^* are strictly convex then the duality mapping $J:X\to X^*$ is maximal monotone with J^{-1} continuous (refer [19]) and hence as a corollary of the above proposition we obtain a very useful result for compact perturbation of the duality mapping.

COROLLARY 3.1. Let X be a strictly convex real reflexive Banach space with strictly convex dual X^*. Let $T:X\to X^*$ be of the type $T=J+C$ where C is compact and continuous with $D(C)=X$. Then T is completely closed.

If we consider the compact and continuous perturbation of the identity mapping we need not assume the strict convexity of X and X^*.

COROLLARY 3.2. Let $T:X\to X$ be of the type $I+C$ where $C:X\to X$ is compact and continuous, then T is completely closed.

Remark 4. The above two corollaries remain valid if, instead of T being defined on the whole space, we assume that it is defined on a closed subset of X.

We are now in a position to give the example of a map which is completely closed but not of type (M).

Example 3.3. Let $X=L_2[-\pi,\pi]$. Define $T:X\to X$ as

$$[Tx](t)=-x(t)+\int_{-\pi}^{\pi}[x(t)-1]^3\,dt.$$

Then T is of the type $-[I+C]$ where C is the compact operator

$$[Cx](t)=\int_{-\pi}^{\pi}[x(t)-1]^3\,dt.$$

Take $x_n(t) = \sin nt$. Then $[Tx_n](t) = [\pi + \sin nt]$. $x_n \to 0$, $Tx_n \to \pi = y$ and $\limsup_n (Tx_n, x_n) \leqslant (y, x)$. But $Tx = -2\pi \neq -\pi = y$.

We now proceed to give the main existence theorem regarding the random solvability of operator equation of the form

$$T(\omega)x(\omega) = y(\omega), \tag{1}$$

where $T:\Omega \times X \to Y$ is a random operator with domain

$$D = D(T) = \bigcap_{\omega \in \Omega} D(T(\omega)),$$

Ω is a complete probability measure space.

THEOREM 3.1. Let $T:\Omega \times D \to Y$ be a completely closed separable random operator such that $\{x: T(\omega)x = y(\omega)\} \neq \phi$ for each $\omega \in \Omega$. Then (1) has a random solution. We presume that each $D(T(\omega))$ is a closed subset of X so that D is closed in X.

Proof. Let $f(\omega) = \{x \in D: T(\omega)x = y(\omega)\}$. Then $f:\Omega \to 2^D$ is a multivalued mapping. We show that this is measurable.

Let $C = S \cap X$, where S is a closed ball in X. We need to show that $f^{-1}(C) \in \beta$. Let $\{x_n\} \subset C$ be a countable dense subset of C. We claim that

$$f^{-1}(C) = \bigcap_{n=1}^{\omega} \bigcup_{x_i \in S} \left\{\omega \in \Omega: \| T(\omega)x_i - y(\omega) \| < \frac{1}{n}\right\}. \tag{2}$$

Let $\omega \in f^{-1}(C)$. This implies that there exists $x \in D$ such that $T(\omega)x = y(\omega)$. Since $\{x_n\}$ is dense in C, separability of T implies that for every integer n there exists $x_{i(n)} \in \{x_n\}$ such that

$$\| x_{i(n)} - x \| < \frac{1}{n} \quad \text{and} \quad \| T(\omega)x_{i(n)} - y(\omega) \| < \frac{1}{n}.$$

This proves that ω belongs to RHS of (2).

Conversely, now assume that ω belongs to RHS of (2). This implies that we have a sequence $\{x_{i(n)}\} \subset C$ such that $T(\omega)x_{i(n)} \to y(\omega)$. As $\{x_{i(n)}\}$ is a bounded sequence in a reflexive Banach space it has a subsequence (which we do not differentiate) converging weakly to x. Thus $x_{i(n)} \rightharpoonup x$ and $T(\omega)x_{i(n)} \to y(\omega)$ and $T(\omega)$ is completely closed, so we get that $x \in D$ and $T(\omega)x = y(\omega)$. This proves (2) and hence the measurability of f.

Also as $T(\omega)$ is completely closed it follows that f has closed values. Thus f is a measurable multivalued mapping with closed values and hence by theorem 2.2 it has a measurable selector $x(\omega)$ which is a random solution of (1). This completes the proof of the theorem.

As simple corollaries of this theorem, we obtain the following known results for random solvability of equation of the type (1) and also of random fixed point equation

$$T(\omega)\, x(\omega) = x(\omega). \tag{3}$$

type (M). Further, let condition (c) be satisfied:

(c) If $\{x_n\}$ is a bounded sequence in X such that for every $\omega\in\Omega$, $\{(T(\omega)x_n, x_n)\}$ is also bounded, then $\{T(\omega)x_n\}$ is bounded for all $\omega\in\Omega$.

Further, assume that there exists an $r>0$ such that $(T(\omega)x, x)\geqslant 0$ for $\|x\|=r$ and all $\omega\in\Omega$. Then there exists a random solution of the operator equation

$$T(\omega)\,x(\omega)=0. \tag{4}$$

Proof. For each $\omega\in\Omega$, $T(\omega):X\to X^*$ is an operator of type (M) and hence by remark 3 it is completely closed. Also, each $T(\omega)$ satisfies all assumptions of proposition 1.2 of [20] and hence it follows that $\{x:T(w)x=0\}\neq\phi$. Thus $T:\Omega\times X\to X^*$ satisfies all hypotheses of theorem 3.1 and so it follows that (4) has a random solution.

COROLLARY 3.4 (Itoh [12]). Let $T:\Omega\times X\to X^*$ be a hemicontinuous coercive monotone separable random operator. Then (1) has a random solution for every random variable $y(\omega)$.

Proof. As hemicontinuous monotone operator defined on the whole space is maximal monotone, it follows by proposition 3.2 that $T(\omega):X\to X^*$ is completely closed for each $\omega\in\Omega$. Also by theorem 2.4 we have $\{x:T(\omega)x=y(\omega)\}\neq\phi$ for each $\omega\in\Omega$. Our result now follows by the application of the main theorem.

COROLLARY 3.5 (Itoh [12]). Let $U:\Omega\times X\to X^*$ be a random operator of the form $U=T+S$ where $T:\Omega\times X\to X^*$ is a demicontinuous monotone separable random operator and $S:\Omega\times X\to X^*$ is a completely continuous random operator. Assume further that U is coercive, then $Ux(\omega)=y(\omega)$ has a random solution for every random variable $y(\omega)$.

Proof. This follows from the Browder's result for deterministic operator equation (refer [7]) and proposition 3.3.

COROLLARY 3.6 (Hans [9]). Let $T:\Omega\times X\to X^*$ be a random contraction. Then there exists a unique random solution of the fixed point equation (3).

Proof. For each $\omega\in\Omega$, $T(\omega):X\to X^*$ is a contraction mapping and hence by Banach contraction mapping theorem it follows that $\{x\in X:[I-T(\omega)]x=0\}\neq\phi$. Also, $T(\omega)$ being a contraction, we get that $[I-T(\omega)]$ is a monotone continuous operator defined on the whole space and hence maximal monotone. So $[I-T(\omega)]$ is a completely closed separable random operator satisfying all hypotheses of theorem 3.1. Hence there exists a random solution $x(\omega)$ of $[I-T(\omega)]x(\omega)=0$, which in turn implies that $x(\omega)$ is a random fixed point of T.

COROLLARY 3.7 (Bharucha Reid [5]). Let E be a closed convex bounded subset of X and let $T:\Omega\times E\to E$ be a compact and continuous random operator. Then T has a

$\omega)x = 0\} \neq \phi$ for each $\omega \in \Omega$. Further, by corollary 3.2 and remark 4 $- T(\omega)]$ is completely closed and hence theorem 3.1 gives the required result.

We now assume that $T:\Omega \times X \to Y$ is an invertible random operator. Then the ndom solvability of the random operator equation $T(\omega)x(\omega) = y$ for every $y \in Y$ is uivalent to the measurability of the inverse operator $T^{-1}:\Omega \times Y \to X$. Theorem 3.1 mediately gives us the following result regarding the measurability of T^{-1}.

EOREM 3.2. Let $T:\Omega \times X \to Y$ be a separable invertible random operator which is mpletely closed. Then $T^{-1}:\Omega \times Y \to X$ is also a random operator.

From this theorem we derive some known measurability inverse theorems.

ROLLARY 3.8 (Nashed and Salehi [18]). Let $T:\Omega \times X \to Y$ be a continuous random erator such that $T(\omega)$ is invertible with $T^{-1}(\omega)$ continuous. Then T^{-1} is also a ndom operator.

ROLLARY 3.9 (Joshi [13]). Let $T:\Omega \times X \to X^*$ be a separable demicontinuous onotone random operator such that $T(\omega)$ is invertible. Then T^{-1} is also a random erator.

ROLLARY 3.10 (Hans [10]). Let $T:\Omega \times X \to Y$ be a bounded invertible random erator. Then T^{-1} is also a random operator.

Random Hammerstein equation

this section we discuss the solvability of random Hammerstein operator equation

$$x(\omega) + K(\omega)N(\omega)x(\omega) = 0, \tag{5}$$

here $N(\omega):X^* \to X$ is a bounded, continuous random operator and $K(\omega):X \to X^*$ is a itable bounded linear random operator. The class of such operator equations cludes a random nonlinear integral equation of the form

$$x(s;\omega) + \int_\Sigma K(s,t;\omega) f(t, x(t;\omega))\, dt = 0.$$

In the deterministic setting, such a type of equation has been extensively studied 8]). In the random setting its study is limited to a few which includes Kannan [15], annan and Salehi [16], Itoh [12] and Joshi [14]. Here we mainly give the random alogs of the results of Amann [1] for (5).

We conclude this section by giving a concrete example.

HEOREM 4.1. Let $K:\Omega \times X \to X^*$ be a compact monotone random operator with nge contained in a closed subspace Y of X^*. Let $N:\Omega \times Y \to X$ be a continuous unded random operator. Assume that there exists a constant $\rho > 0$ such that for all

Proof. By theorem 1 of Amann [1], $\{x: x + K(\omega)N(\omega)x = 0\} \neq \phi$. As $K(\omega)$ is bounded and random and $N(\omega)$ is random, it follows that $K(\omega)N(\omega)$ is a random operator. Also continuity of both of these operators gives the separability of $K(\omega)N(\omega)$. Thus for each $\omega \in \Omega$, $T(\omega) = [I - K(\omega)N(\omega)]$ is of the type $I + C(\omega)$, where $C(\omega)$ is a compact and continuous operator and hence by corollary 3.2 it follows that $T(\omega)$ is completely closed. Theorem 3.1 gives a random solution $x(\omega)$ of (5) with $\|x(\omega)\| \leqslant \rho$.

If we assume that $K(\omega): X \to X^*$ is angle-bounded, we can considerably relax the condition imposed on $N(\omega)$. We first state without proof the following proposition which is a random analog of the splitting lemma for compact angle-bounded linear operators $K: X \to X^*$ (refer Browder [8]).

PROPOSITION 4.1. Let X be a real separable Banach space and let $K: \Omega \times X \to X^*$ be a compact and monotone angle-bounded operator with $\|K(\omega)\| \leqslant K_0$ and with a fixed constant c of angle-boundedness for all $\omega \in \Omega$. Then there exists a separable real Hilbert space H (independent of ω) and a compact linear random operator $S: \Omega \times X \to H$ and a skew symmetric bounded linear operator $B: \Omega \times H \to H$ such that for each $\omega \in \Omega$

$$K(\omega) = S^*(\omega)(I + B(\omega))S(\omega),$$

and

$$\|S(\omega)\|^2 \leqslant K_0, \quad \|B(\omega)\| \leqslant c$$

and

$$[[I + B(\omega)]^{-1}h, h]_H \geqslant (I + c^2)^{-1}\|h\|_H^2 \quad \text{for all } h \in H.$$

THEOREM 4.2. Let X and K be as in proposition 4.1 with $R(K(\omega))$ contained in a closed subspace Y of X^* for all $\omega \in \Omega$. Let $N: \Omega \times Y \to X$ be a continuous and bounded random operator and assume that there exists a function $\phi: R_+ \to R_+$ satisfying $\phi(\rho) = O(\rho^2)$ as $\rho \to \infty$ such that for all $x \in Y$

$$(x, N(\omega)x) \geqslant -\phi(\|x\|) \quad \text{for all } \omega \in \Omega.$$

Then (5) has a random solution in Y.

Proof. The solvability of equation (5) is equivalent to the solvability of

$$T(\omega)x(\omega) = 0 \tag{6}$$

in the Hilbert space H where $T(\omega) = [I + B(\omega)]^{-1} + S^*(\omega)N(\omega)S(\omega)$.

First observe that $\{x: T(\omega) = 0\} \neq \phi$ for each $\omega \in \Omega$ (Theorem 3 of Amann [1]). Further, as $S(\omega)$ is a compact linear random operator and $N(\omega)$ is a bounded continuous random operator, it follows that $S^*(\omega)N(\omega)S(\omega)$ is a compact and continuous random operator. Also $[I + B(\omega)]^{-1}$ is a bounded linear random operator (corollary 3.10 of [14]). Hence it follows by proposition 3.4 that $T(\omega)$ is completely closed and consequently theorem 3.1 gives a random solution of (6).

Example 4.1. Consider a nonlinear random Sturm-Liouville's problem

$$\frac{d}{dt}(p(t)x(t)) + q(\omega; t)x(t) + f(\omega; t, x(t)) = 0$$

$$x(0) = 0 = x(1).$$

We assume that q is a random function. Likewise f is also a random function def on $[0, 1] \times R$ satisfying Caratheodory conditions and a growth condition of the

$$|f(\omega; t, x)| \leqslant a(t) + b(\omega)|x|,$$

where $b(\omega) > 0$ is a random constant and $a \in L_2[0, 1]$. We further assume that t exists a constant $\rho > 0$ such that

$$xf(\omega; t, x) \geqslant 0 \quad \text{for } |x| > \rho \quad \text{for all } \omega \in \Omega.$$

Assume that the Green's function $K(\omega; s, t)$ of the system (7) exists and is of Hilb Schmidt type for each $\omega \in \Omega$. Also assume that $K(\omega; s, t)$ is symmetric and monot Then (7) is equivalent to the Hammerstein integral equation

$$x(\omega; s) + \int_0^1 K(\omega; s, t) f(\omega; t, x(\omega; t))\, dt = 0.$$

Define random operators $K(\omega)$, $N(\omega)$ on $L_2[0, 1]$ as

$$[K(\omega)]x(s) = \int_0^1 K(\omega; s, t)x(t)\, dt$$

$$[N(\omega)]x(s) = f(\omega; s, x(s)).$$

Then the solvability of (6) is equivalent to the solvability of (8). To apply theo 4.2 we set $Y = X = X^* = L_2[0, 1]$.

As in Amann [1] it is easy to check the conditions of the above theorem and he we get a random solution of (7), with relatively mild conditions on the nonlir function f.

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Melting of a long cylinder due to an axisymmetric hot spot

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Abstract. By employing a new embedding technique, a short-time analytical solution for the melting of a long cylinder due to an axisymmetric hot spot is presented in this paper. The melt is removed as soon as it is formed. The initial propagation of the melt along the interior of the cylinder exhibits a well-known behaviour. However, the melt propagation along the surface of the cylinder is of a new type and depending on the applied flux whether it is the continuation of the pre-melting flux or different, the initial propagation of the melt along the surface is proportional to $t^{1/2}$ or $t^{1/4}$, where t is the dimensionless time. The melt propagation is much faster along the surface of the cylinder than towards the interior.

Keywords. Embedding technique; hot spot; ablation; moving boundary problems; isotherm condition.

1. Introduction

The melting problem* studied in this paper can be put under a more general class of problems known as moving boundary problems (MBPs). A typical feature of the MBPs is that apart from the fixed boundary of the region under consideration, there is also a boundary inside the region which is unknown and is moving with time and some conditions are to be satisfied on this moving boundary. These MBPs arise in many physical situations and their utility and some of the methods of analytical and numerical solutions can be found in [16]. Depending on the portion of the fixed boundary on which the melting initiates, Sikarskie and Boley [19], have divided the melting problems into three classes. In class I problems, the melting starts simultaneously at all points on the fixed boundary. In classes II and III problems, the melting starts over a portion of the boundary and at a point of the boundary respectively. In this paper, we are mainly concerned with a class III problem and the solution of class I problem will be derived as a particular case of the general problem.

Because of the non-linearity of melting problems and the unknown nature of the moving boundary, their exact analytical solutions valid for total melting time and varied shapes are rather unconceivable. As compared to class III problems, class I problems are easier to tackle. In the latter case, only one boundary condition is to be satisfied over the entire boundary of the solid (say, when flux is prescribed all over the

* The solidification problem is mathematically analogous to the melting problem and it is sufficient to discuss the melting problem here.

boundary of the solid) whereas in the former case, at least two boundary conditions are to be satisfied as till the melting is completely over, the melt spread will cover only a portion of the fixed boundary. Further, class III problems have to be at least two-dimensional.

For one-dimensional class I melting problems, some exact similarity solutions exist in the literature such as Neumann solution [8], the series solution by Tao [20], the solution by Ozisik and Uzzell [17] etc. Some more references can be found in the above mentioned references. In view of the difficulty of the exact analytical solutions, attempts have been made to obtain short-time analytical solutions which are of both mathematical and practical interest. Boley *et al* [4] have made significant contributions in the development of short-time solutions of classes I, II and III problems. Some of these class I short-time solutions have been extended numerically to larger times by employing suitable numerical schemes [14].

Recently Gupta and Lahiri [12] and Gupta [11] have obtained short-time solutions of class I problems pertaining to cylindrical molds. We shall not discuss here in detail the methods employed for the solution of class I problems such as similarity solutions, perturbation methods, approximate methods etc. for the simple reason that none of the above methods are suitable for class III problems and further for multi-dimensional problems, sufficient applications of these techniques are not available in the literature.

Class III problems arise in many physical situations such as resistance spot welding, laser welding, solidification of castings resulting from any unavoidable initial eveness in the mold or due to uneveness in the ablation of heat shields etc. In the laser welding process the interaction time is generally less than 0·1 sec and the melt depth is generally less than 1 mm [13]. Short-time analytical solutions are ideally suited for direct applications or simulation of experimental runs in such processes. Short-time solutions can be used for starting some of the numerical schemes like Murray and Landis scheme [15] and enthalpy scheme [9].

On class III problems only three works [5–7] by Boley and co-workers exist in the literature which pertain to short-time solutions of one phase ablation problems. Out of these three works, the one by Yagoda and Boley [6] is more relevant to the present work in which an axisymmetric melting in the half-space (and also the planar problem) due to hot spot has been studied. The prescribed flux is constant and the melt is removed as soon as it is formed. The problem considered is symmetric in radial coordinate so that it matches the planar case. In the present paper, the problem considered pertains to a long cylinder and the problem is symmetric only in axial coordinate. The prescribed flux is time- and space-dependent in the present paper. The embedding technique given by Boley [2] for class I problems was suitably adapted to handle the class III problems in [5–7]. Some comparison of the Boley's embedding technique and the embedding technique employed in the present paper will be in order. Boley's embedding technique consists of prescribing fictitious flux in the case of ablation problems and fictitious boundary flux (or fictitious boundary temperature) together with suitable fictitious initial temperature in the case of solidification

determined by repeated differentiations of the equations with respect to the time variable and taking appropriate limits. This differentiation yields considerable simplification in the determination of unknowns and the approach becomes straightforward. This differentiation of integrals is not possible in Boley's technique. For the class III problems in [5–7], Boley's embedding technique was used in conjunction with a semi-inverse approach. This semi-inverse approach consists of assuming suitable expressions for fictitious flux, melt spread along the surface and melt depth towards the interior etc and the unknown quantities are determined by equating the various powers of the different variables on both sides of the equations. The approach in the present paper is direct and more general. Some more features will be discussed after obtaining the solution.

2. Problem formulation

A long cylindrical solid occupies the region $0 \leqslant r \leqslant R_0$, $|z| < \infty$ at time $t = 0$ (axisymmetric problem). This solid is heated and the prescribed flux $\hat{Q}(z, t)$ on the surface $r = R_0$ of the cylinder is known. The temperature of the body at $t = 0$ is a known quantity, $\hat{f}(r, z)$, which is symmetric in z and is such that

$$\begin{aligned} \hat{f}(r, z) &= T_m, \quad z = 0, \quad r = R_0 \\ &< T_m, \quad z \neq 0, \quad r \neq R_0, \end{aligned} \tag{1}$$

and further

$$(\partial \hat{f}/\partial z) = 0, \quad z = 0; \quad (\partial^2 \hat{f}/\partial z^2) \neq 0 \quad \text{at } r = R_0 \text{ and } z = 0. \tag{2}$$

T_m is the unique melting temperature. The melting will initially start at $r = R_0$ and $z = 0$ and with time it will propagate along the surface of the cylinder and towards the interior of the solid. It is assumed that the melt is removed as soon as it is formed. The equation of the melting front may be written in the following mathematical form

$$r = s(z, t); \quad |z| \leqslant b(t), \tag{3}$$

where $|z| = b(t)$ is the equation of the melt propagation along the surface of the cylinder. $s(z, t)$ and $b(t)$ are unknowns and are to be determined together with the temperature in the solid by solving the following dimensionless differential equation, initial and boundary conditions

$$2\frac{\partial T}{\partial V} = V\left(\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} + \frac{\partial^2 T}{\partial Z^2}\right), \tag{4}$$

$$0 \leqslant R \leqslant S(Z, V), \quad |Z| < \infty, \quad V > 0,$$

$$T(R, Z, 0) = f(R, Z) \tag{5}$$

$$\left.\frac{\partial T}{\partial R}\right|_{R=1} = Q(Z,V), \quad |Z| > B(V), \tag{6}$$

$$T(S,Z,V) = 1, \quad |Z| \leqslant B(V), \tag{7}$$

$$\left[1+\left(\frac{\partial S}{\partial Z}\right)^2\right]\left.\frac{\partial T}{\partial R}\right|_{R=S} = Q(Z,V) + \frac{2\lambda}{V}\frac{\partial S}{\partial V}, \tag{8}$$

$$|Z| \leqslant B(V),$$

$$B(V)|_{V=0} = 0, \tag{9}$$

$$S(Z,V)|_{V=0} = 1, \tag{10}$$

$$S(Z,V)|_{|Z|=B(V)} = 1. \tag{11}$$

$T(R,Z,V)$ is the dimensionless temperature. All the temperatures have been made dimensionless with the help of the melting temperature T_m. Equation (7) is the isotherm condition and (8) is the heat balance condition. In (8) it has been assumed that the resultant heat flux vector is along the direction of R and (8) can be deduced with the help of the results given in [18]. The following dimensionless variables have been used in (4)–(11).

$$Z = z/R_0, \quad R = r/R_0, \quad V = 2(kt/R_0^2)^{1/2},$$

$$\lambda = l/cT_m, \quad f(R,Z) = \hat{f}(r,z)/T_m, \tag{12a}$$

$$Q(Z,V) = Q(z,t)\cdot R_0/KT_m, \quad S(Z,V) = s(z,t)/R_0,$$

$$B(V) = b(t)/R_0, \tag{12b}$$

where R_0 is the radius of the cylinder, K the thermal conductivity, k the diffusivity, l the latent heat and c the specific heat of the solid. Thermal properties are taken to be constants. $Q(Z,V)$ is symmetric in Z.

2.1 *Pre-melting temperature*

The function $\hat{f}(r,z)$ in (1) is essentially the pre-melting temperature and it can be either prescribed directly or obtained from the solution of a heat conduction problem without phase change with (say) flux prescribed boundary condition. We shall avoid here the solution of this pure heat conduction problem and simply mention that $\hat{f}(r,z)$ (if it is not prescribed) will be the temperature of the solid cylinder at some time $t = t_m$, $t_m > 0$, and t_m in class III problem may be obtained by using the condition (1). The dimensionless time V in this case can be redefined as

$$V = 2[k(t-t_m)/R_0^2]^{1/2}. \tag{13}$$

The quantity

$$\frac{KT_m}{}\left.\frac{\partial f}{}\right|$$

can also be obtained by the present method of solution.

3. Solution

In view of the boundary conditions (6)–(8), the temperature solution will be written in the following form

$$T(R, Z, V) = T_{\mathrm{I}}, \quad 0 \leqslant R < \infty, \quad |Z| > B(V), \tag{14}$$

$$= T_{\mathrm{I}} + T_{\mathrm{II}}, \quad 0 \leqslant R < \infty, \quad |Z| \leqslant B(V), \tag{15}$$

$$T_{\mathrm{I}} = \frac{2}{\sqrt{\pi V^3}} \left[\int_{-\infty}^{\infty} \int_{0}^{1} H(p, q) f(p, q)\, \mathrm{d}p\, \mathrm{d}q + \int_{-\infty}^{\infty} \int_{1}^{\infty} H(p, q) f_2(p, q)\, \mathrm{d}p\, \mathrm{d}q \right], \tag{16}$$

$$T_{\mathrm{II}} = \frac{2}{\sqrt{\pi V^3}} \int_{-\infty}^{\infty} \int_{1}^{\infty} H(p, q) f_3(p, q)\, \mathrm{d}p\, \mathrm{d}q, \tag{17}$$

$$H(p, q) = p \exp\left[- \{R^2 + p^2 + (Z - q)^2\}/V^2 \right] I_0(2Rp/V^2). \tag{18}$$

$I_0(x)$ is the modified Bessel function of the first kind of order zero.

The above representation of $T(R, Z, V)$ in (14) and (15) needs some clarification. Once the melting starts, the boundary of the solid for $|Z| > B(V)$ is represented by $R = 1$ and for $|Z| < B(V)$ it is unknown. The boundary conditions are different on these two different portions of the solid boundary and there does not seem to be any systematic way to construct even short-time solution in which the temperature satisfies all the boundary conditions and the melting in the region $|Z| < B(V)$ affects the temperature in the region $|Z| > B(V)$. In view of this extremely difficult situation, we have tried to construct a short-time solution with the assumption that the temperature in the region $|Z| > B(V)$ is interacting with the temperature in the region $|Z| < B(V)$ but the temperature in the region $|Z| < B(V)$ is not affecting the temperature in the region $|Z| > B(V)$. However, T_{I} and T_{II} both satisfy the boundary conditions given by (6)–(8). We shall see later (figure 1) that for short time this assumption is reasonably justified. The error is insignificant and is confined to a very small neighbourhood of the point $R = 1$, $|Z| = B(V)$. It seems relevant to point out here that in [5–7] also a similar assumption has been made but not explicitly mentioned (for example refer equation (11) in [6]).

It can be easily checked that T_{I} and T_{II} both satisfy (4) and T_{I} satisfies (5). $f_2(R, Z)$ is unknown and is the initial temperature in the fictitious extension $1 \leqslant R < \infty$, $|Z| < \infty$ of the original region. $f_3(R, Z)$ is an unknown function introduced for mathematical convenience.

Mathematically, for the determination of three unknowns, namely, $S(Z, V)$, $f_2(R, Z)$ and $f_3(R, Z)$, there are three conditions (6) to (8). From (11), $B(V)$ can be determined. We shall first formally obtain the short-time solution and then try to justify some of the

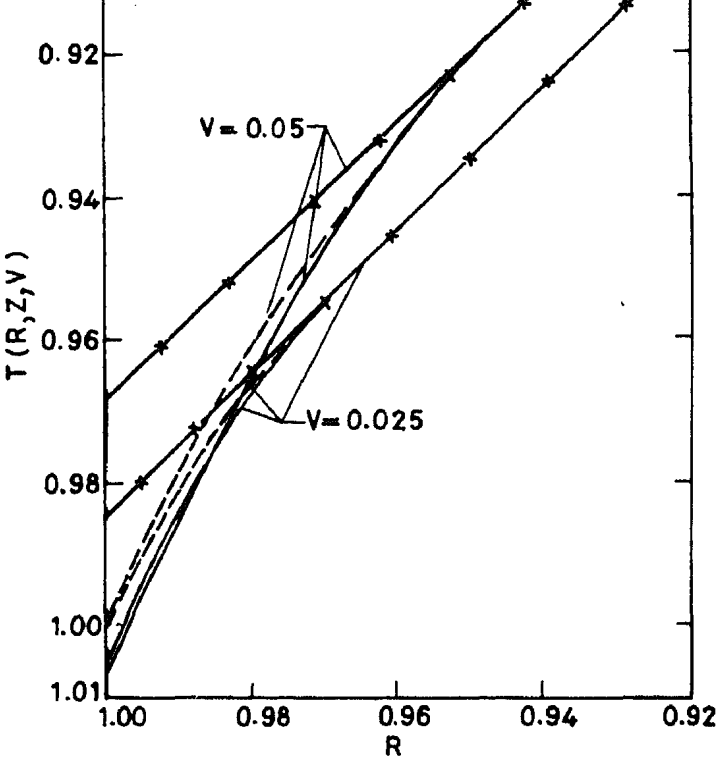

Figure 1. Comparison of initial temperature $f(R,Z)$ and the temperatures T_{I} and T_{II}[(16) and (17)] at $Z=B(V)$ and different values of R. $Q(Z,V)=2\exp(-Z^2)$ and $f(R,Z)=R\{1-5Z^2\exp(-Z^2)\}$ for $V=0{\cdot}025$ and $0{\cdot}05$ and $B(V)=0{\cdot}0569$ and $0{\cdot}0796$ respectively. The curves — × —, — and --- are for f, T_{I} and T_{II} respectively. $\lambda=0{\cdot}6928$.

be used for $I_0(x)$

$$I_0(x)=\frac{e^x}{(2\pi x)^{1/2}}\left(1+\frac{1}{8x}+\frac{9}{128x^2}+\cdots\right). \tag{19}$$

Firstly, in (16) and (17) $I_0(x)$ is replaced by the first two terms of the asymptotic series in (19) and then these temperature solutions are substituted in (6)–(8). After making appropriate substitutions, the following equations are obtained.

$$\begin{aligned}-\pi VQ(Z,V)=&\int_{\infty}^{-\infty}\int_{1/V}^{0}D_1(V,p)f_1(1-Vp,Z-Vq)\\&\times\exp\{-(p^2+q^2)\}\,dp\,dq\\&+\int_{\alpha}^{-\infty}\int_{0}^{-\infty}\{D_1(V,p)/(1-Vp)^{1/2}\}f_2(1-Vp,Z-Vq)\\&\times\exp\{-(p^2+q^2)\}\,dp\,dq,\quad |Z|<\infty,\end{aligned} \tag{20}$$

$$\pi S^{3/2}=\int_{\infty}^{-\infty}\int_{S/V}^{(S-1)/V}D_2(S,V,p)f_1(S-Vp,Z-Vq)$$

$$\times \exp\{-(p^2+q^2)\}\,dp\,dq$$

$$+\int_{\infty}^{-\infty}\int_{(S-1)/V}^{-\infty}\{D_2(S,V,p)/(X-Vp)^{1/2}\}$$

$$\times\{f_2(S-Vp,Z-Vq)+f_3(S-Vp,Z-Vq)\}$$

$$\times \exp\{-(p^2+q^2)\}\,dp\,dq, \quad |Z|\leqslant B(V), \tag{21}$$

$$\left\{1+\left(\frac{\partial S}{\partial Z}\right)^2\right\}\left[\int_{\infty}^{-\infty}\int_{S/V}^{(S-1)/V} D_3(S,V,p)f_1(S-Vp,Z-Vq)\right.$$

$$\times \exp\{-(p^2+q^2)\}\,dp\,dq+\int_{\infty}^{-\infty}\int_{(S-1)/V}^{-\infty}$$

$$\times\{D_3(S,V,p)/(X-Vp)^{1/2}\}\{f_3(S-Vp,Z-Vq)$$

$$\left.+f_2(S-Vp,Z-Vq)\}\exp\{-(p^2+q^2)\}\,dp\,dq\right]$$

$$=\pi VQ(Z,V)+2\pi\lambda(\partial S/\partial V), \quad |Z|\leqslant B(V), \tag{22}$$

where

$$D_1=2p[(1-Vp+(V^2/16))]+[V(1-Vp)/2]+3V^3/32, \tag{23}$$

$$D_2=S(S-Vp)+(V^2/16), \tag{24}$$

$$D_3=2p\{S^2(S-Vp)+(V^2S/16)\}+[VS(S-Vp)/2]+(3V^3/32), \tag{25}$$

$$f_1(R,Z)=f(R,Z)/\sqrt{R}. \tag{26}$$

In (19), (6) has been extended to $|Z|<\infty$ and this is permissible as at $R=1, |Z|<B(V)$ no boundary condition is prescribed.

The following series expansions for the various known and unknown functions will be assumed

$$f_1(R,Z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(R-1)^m}{\lfloor m}\frac{Z^n}{\lfloor n}\frac{\partial^{m+n}(f_1)}{\partial R^m\partial Z^n}\bigg|_{\substack{R=1\\Z=0}},$$

$$0\leqslant R\leqslant 1, \quad |Z|<\infty \tag{27}$$

$$f_i(R,Z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(R-1)^m}{\lfloor m}\frac{Z^n}{\lfloor n}\frac{\partial^{m+n}(f_i)}{\partial R^m\partial Z^n}\bigg|_{\substack{R=1\\Z=0}},$$

The series expansions in (27) and (28) are motivated by the fact that in a short-time solution the behaviour of the functions near the point of initiation of the melt is more important. Without any loss of generality, $Q(Z, V)$ can be represented by (29). $S(Z, V)$ is motivated by the Neumann solution and the series for $B(V)$ is tentative and can be adjusted according to the details of (11). It will be assumed that the series expansions in (27) and (28) are uniformly convergent in the regions indicated there and the series expansions in (29)–(31) are uniformly convergent in some region $|Z| < Z_0, 0 \leqslant V < V_1$. These assumptions will be further discussed later. In order to obtain the solution, the series expansions given in (27)–(30) are substituted in (20)–(22) and then the limits $V \to 0+$ of (20)–(22) are taken. Three equations in three unknowns are obtained which when solved give unique solution. Equations (20)–(22) are then differentiated once with respect to V and limit $V \to 0+$ is taken. Again three equations in three unknowns are obtained which when solved give unique solution. This process of higher order differentiations and limits $V \to 0+$ can be continued till the desired coefficient in the series (30) of the moving boundary is obtained. However, after few differentiations the algebra becomes very complicated. $A_1(Z)$ is the root of the transcendental equation (32)

$$\sqrt{\pi\lambda} A_1(Z) = \exp(-A_1^2)\{1 - f_1(1, Z)\}/(1 + \operatorname{erf} A_1), \quad |Z| \leqslant B(V) \tag{32}$$

$$f_2(1, Z) = f_1(1, Z), \tag{33}$$

$$f_3(1, Z) = 2\{1 - f_1(1, Z)\}/(1 + \operatorname{erf} A_1). \tag{34}$$

$A_2(Z)$, $A_3(Z)$ and other related quantities are given in the appendix. In principle, other coefficients $A_4(Z)$, $A_5(Z)$ etc can also be determined but as mentioned earlier, the algebra becomes extremely complicated. It may be checked from (32) that $A_1(0) = 0$ and for $|Z| > 0$, $A_1(Z) > 0$. It is this property of $A_1(Z)$ which enables us to determine $B(V)$. From (34) it is clear that $f_3(1, 0) = 0$ which it should be otherwise the limits of $T(R, 0, 0)$ as $R \to 1+$ and $R \to 1-$ will not be the same.

Till now $B(V)$ has not come into the picture and the method of solution is tentative upto this stage as we do not know whether for a given value of V ($V \ll 1$), there exists a value of Z or not such that $S(Z, V) = 1 + \sum_{n=1}^{3} A_n(Z, V)^n = 1$. If such a value of Z does not exist then the method of solution fails for the hot spot problem. As we shall see later that the method of solution works and a unique value of Z is obtained. To determine $B(V)$, the condition (11) will be used in the following form

$$\begin{aligned} &A_1''(0)(b_1 V^{1/2} + b_2 V + b_3 V^{3/2} + \cdots)^2 V/2 \\ &+ A_1''''(0)(b_1 V^{1/2} + b_2 V + b_3 V^{3/2} + \cdots)^4 V/24 \\ &+ \{A_2(0) + (b_1 V^{1/2} + b_2 V + b_3 V^{3/2} + \cdots)^2 A_2''(0)/2\} V^2 \\ &+ A_3(0) V^3 + \cdots = 0 \quad \text{for all } V \geqslant 0. \end{aligned} \tag{35}$$

$Z = 0$. The following two cases arise.

(i) $A_2(0) \neq 0$ i.e. the pre-melting flux and post-melting flux are not equal. In this case (35) gives

$$b_1^2 = -2A_2(0)/A_1''(0), \quad A_1''(0) \neq 0; \quad b_2 = 0 \tag{36}$$

$$b_3 = -\{24A_3(0) + 12b_1^2 A_2''(0) + b_1^4 A_1''''(0)\}/[24b_1 A_1''(0)] \tag{37}$$

$$A_2(0) = \frac{1}{4\lambda}\left\{\left.\frac{\partial f}{\partial R}\right|_{\substack{R=1\\Z=0}} - Q_1(0)\right\}, \tag{38}$$

$$A_1''(0) = -\left.\frac{1}{\sqrt{\pi\lambda}}\frac{\partial^2 f_1}{\partial Z^2}\right|_{\substack{R=1\\Z=0}}, \tag{39}$$

$$\sqrt{\pi}A_1''''(0) = 12(A_1''(0))^2 - \left.\frac{1}{\lambda}\frac{\partial^4 f_1}{\partial Z^4}\right|_{\substack{R=1\\Z=0}}, \tag{40}$$

$$A_2''(0) = \frac{1}{8\lambda}\left[\left.\frac{\partial^2 f_1}{\partial Z^2}\right|_{\substack{R=1\\Z=0}} - 2\sqrt{\pi}A_1''(0)\left.\frac{\partial f_1}{\partial R}\right|_{\substack{R=1\\Z=0}} - 4Q_1''(0) + 2\left\{\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\left(\left.\frac{\partial f_1}{\partial R}\right|_{R=1}\right)\right\}_{Z=0}\right], \tag{41}$$

$$A_3(0) = \frac{1}{12\sqrt{\pi\lambda}}\left[-2\sqrt{\pi}Q_2(0) - 4(2A_2(0)+1)\left.\frac{\partial f_1}{\partial R}\right|_{\substack{R=1\\Z=0}} + 33/8 - [5A_2(0)/2] - 2\left.\frac{\partial^2 f_1}{\partial R^2}\right|_{\substack{R=1\\Z=0}} - \left.\frac{\partial^2 f_1}{\partial Z^2}\right|_{\substack{R=1\\Z=0}}\right]. \tag{42}$$

It can be easily shown that b_1 is real which it should be and this provides a very useful check on the method of solution. It may be remarked here that the melt propagations along the surface and towards the interior have to be investigated separately. The melt propagation along the surface is proportional to $V^{1/2}$, $V \ll 1$. If

$$\left.\frac{\partial^2 f_1}{\partial Z^2}\right|_{\substack{R=1\\Z=0}} = 0$$

and

$$\left.\frac{\partial^4 f_1}{\partial Z^4}\right|_{\substack{R=1\\Z=0}} \neq 0,$$

then $B(V)$ is proportional to $V^{1/4}$, $V \ll 1$. In this case we modify (31) and write

$$\sum_{n=0}^{\infty}

known. In the previous works also [21], it was observed that the sign of A_3 can be shown to be negative only if $f(R,Z)$ is given and not otherwise. $B(V)$ is proportional to V, $V \ll 1$. If

$$\left.\frac{\partial^2 f_1}{\partial Z^2}\right|_{\substack{R=1\\Z=0}} = 0 \quad \text{and} \quad \left.\frac{\partial^4 f_1}{\partial Z^4}\right|_{\substack{R=1\\Z=0}} \neq 0,$$

then $B(V)$ is proportional to $V^{1/2}$, $V \ll 1$. b_1 in (36) and b_2 in (43) are independent of material parameters but b_3 in (37) depends on the material parameter. It may be mentioned here that even if higher order terms like V^4, V^5 etc. are considered in (35), b_1, b_2 and b_3 given above will remain unchanged. For all practical purposes, $S(Z,V)$ is given by

$$S(Z,V) = 1 + [A_1''(0)Z^2V/2] + A_1''''(0)Z^4V/24 + A_2(0)V^2 + [A_2''(0)Z^2V^2/2] + A_3(0)V^3. \tag{44}$$

For $V \ll 1$, the leading term in (44) is not $A_1''(0)Z^2V/2$ as $A_1''(0)$ is always positive and this will mean that for $V \ll 1$, $S(Z,V) > 1$. For this, we write

$$S(Z,V) = 1 + A_2(0)V^2\left(1 - \frac{Z^2}{b_1^2 V}\right) + \cdots. \tag{45}$$

$A_2(0)$ is always negative (refer (38)) and $Z^2/b_1^2V < 1$ ($|Z| < B(V)$) and therefore melt propagation along the interior is proportional to V^2. Similarly, if $A_2(0) = 0$, the melt propagation towards the interior is proportional to V^3. These behaviours of the moving boundary are well known [19]. However the shape of the moving boundary in (45) is new in the literature. Along with the unknowns in the moving boundary, the unknowns in the temperature solutions in (14) and (15) are also determined. Only the temperature distribution T_{II} is being given here as T_I can be easily obtained from the expression of T_{II} after making a minor change. We replace $I_0(x)$ in (17) by the first two terms of the asymptotic series in (19), use series expansions for f_1, f_2 and f_3 given in (27) and (28) and integrate term by term. For $V \ll 1$, $0 \leqslant R \leqslant S(Z,V)$ and $|Z| < B(V)$, T_{II} is given by

$$T_{II} = -\tfrac{1}{2}\left\{\sqrt{R}f_1(1,Z) + \sqrt{R(R-1)}\left.\frac{\partial f_1}{\partial R}\right|_{R=1}\right\}\operatorname{erfc}\{(R-1)/V\}$$
$$+\frac{1}{2\pi}\left\{-\sqrt{R}V\left.\frac{\partial f_1}{\partial R}\right|_{R=1} - \frac{V}{\sqrt{R}}f_1(1,Z)\right\}\exp\{-(R-1)^2/V^2\}$$
$$+\tfrac{1}{2}\left\{f_2(1,Z) + f_3(1,Z) + (R-1)\left.\left(\frac{\partial f_2}{\partial R} + \frac{\partial f_3}{\partial R}\right)\right|_{R=1}\right\}$$

$$\times [1 + \operatorname{erf}\{(R-1)/V\}] + \frac{V}{4\sqrt{\pi R}}\left[f_2(1,Z) + f_3(1,Z) \right.$$

$$\left. + 2R\left(\frac{\partial f_2}{\partial R} + \frac{\partial f_3}{\partial R}\right)\Big|_{R=1}\right] \exp\{-(R-1)^2/V^2\}$$

$$+ \text{terms of the type } (R-1)^m V^n,$$

$$\text{where } m+n \geqslant 2; \quad V > 0. \tag{46}$$

In obtaining the above equation, whenever the limit of integration is R/V, it is taken as infinity and this is justified as the integrals are error functions integrals [1]. T_1 can be obtained from (46), by putting $f_3(R, Z) = 0$. T_1 can be used only for $|Z| > B(V)$. It may be noted that in (46), $|R-1|$ and $|Z|$ can take only small values.

3.1 *Heat conduction without phase change*

Short-time solution of the pure heat conduction problem discussed in §2.1 with $f(R, Z)$ as the prescribed initial temperature and $Q(Z, V)$ as the prescribed flux is also given by T_1 which can be used for small values of the quantities V, $|R-1|$ and $|Z|$.

3.2 *Class I problem*

If $f(1, Z) = 1$ for $|Z| < \infty$ and the prescribed flux is positive, then the melting will start instantaneously and simultaneously at all points of the surface $R = 1$. T_1 and T_{II} have to be identical in this case and so $f_3(R, Z) = 0$. Therefore by putting $f(1, Z) = 1$ and $f_3(R, Z) = 0$ in the results given earlier, the solution of class I problem can be obtained. If the formulation similar to the present one (with appropriate changes) is done for Neumann Problem ([8], p. 285) then the moving boundary and temperature distributions in the solid and liquid regions are obtained exactly. This provides a useful check on the present method of solution for class I problems.

4. Discussion and comparison with earlier works

To save space, only few important points are being discussed below:

(i) The rigorous justification of some of the steps in the method of solution and the utility of the assumptions made earlier about the uniform convergence of the series expansions given in (27)–(31) can be discussed on the same lines as done in [11] and [10] in which class I problems have been studied.

(ii) If $f_1(R, Z)$ is given by (26) then the integrands in (20)–(22) have no singularities. Equation (26) for defining $f_1(R, Z)$ can be suitably modified if more than two terms of the series in (19) are to be retained. However, it may be noted that the asymptotic series in (19) does not converge.

(iii) The results given in (36), (37) and (43) are new in the literature. As mentioned earlier, the problems considered in [6] and the present paper are different. The moving

differences mentioned earlier in the techniques there seems to be a fundamental difference in the solution procedure in [6] and the present paper. In the present paper the temperature T_1 in (16) is a function of both odd and even powers of V (V is equal to $y^{1/2}$ multiplied by a constant in [6] and T_1 has the same role as that of T^* (Eq. (11) in [6]) and both are the solutions of pure heat conduction problems) whereas in [6], T^* has been expanded in a series of powers of $y(V^2)$ and not $y^{1/2}$. As a result of this the lowest order term in y on right side of (15a) in [6] is y whereas in the present paper, the lowest order term (other than a constant) in time variable in T_1 is V. Without any loss of generality, T^* in [6] could have been expanded in a series of powers of $y^{1/2}$ and if for some reasons, only even or odd powers of $y^{1/2}$ should occur, then the mathematical solution should automatically bring out these facts as it does in the present work. In this respect the present solution procedure is much more general.

5. Numerical work

There does not seem to be any systematic way to find analytically the radii of convergence of the series for $S(Z, V)$ and $B(V)$ given in (30) and (31) respectively. However, by calculating numerically the coefficients in the series for $B(V)$, a good estimate of the time for which the present solution is valid can be determined. From the large number of numerical experiments carried out in the earlier works on class I problems in [11], [21] and [10] (in [21], for one-dimensional radially symmetric problems, the short-time analytical solution has been checked with the numerical schemes also), a reasonably good criterion has been developed for determining the time for which a short-time solution is valid. For the class III problem in the present work, the criterion given below for the time for which the present solution is valid is motivated by the earlier works and the numerical experiments carried out for the present work.

Consider the series for $B(V)$ given in (31). If the absolute values of the coefficients b_1, b_2, b_3 etc go on decreasing (for given set of parameter values), then by calculating $|b_n V^{n/2}|$ for a given V (where b_n is the last coefficient calculated in $B(V)$) it can be found out whether $|b_{n+1} V^{(n+1)/2}|$ will contribute significantly towards the calculation of $\sum_{m=1}^{n} b_m V^{m/2}$ or not (it is assumed here that the absolute values of the coefficients b_{n+1}, b_{n+2}, etc will also go on decreasing) and this will determine the range of time for which the solution is valid. If only one coefficient is available then V can be taken arbitrarily very small. The numerical results presented in figures 1–3 justify this criterion.

For class I problems in the earlier works mentioned above, this type of criterion for the series $\sum_{m=1}^{n} A_m V^m$ can be discussed with the help of the coefficients A_1, A_2, A_3 etc but since A_1 is of different nature in the present work (for example in the class I problem $A_1(Z)$ will be negetive for all Z), we had to evolve the criterion in terms of the coefficients b_1, b_2, etc.

In figure 1, the effect of melting in the region $|Z| < B(V)$ on the temperature in the region $|Z| > B(V)$ is shown. This effect will be maximum at $|Z| = B(V)$ and therefore the

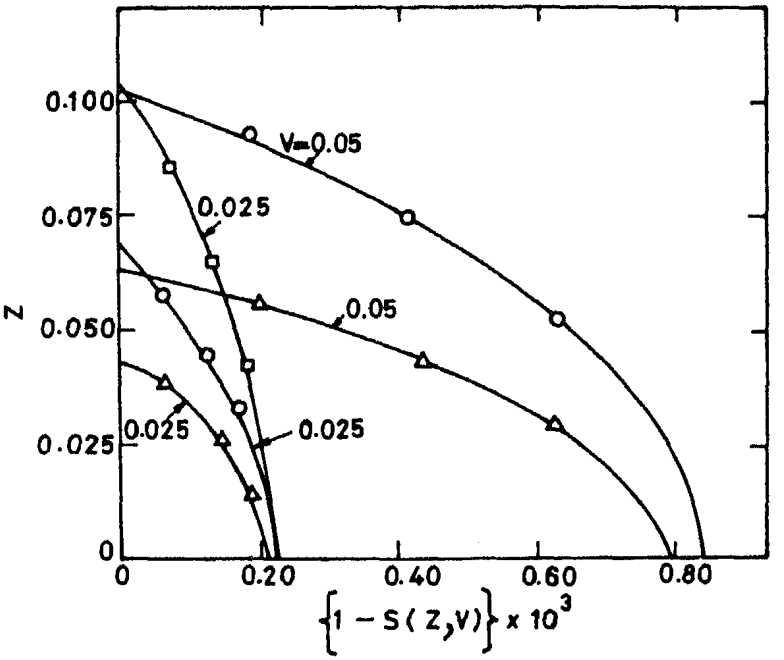

Figure 2. Effect of initial temperature on the melting front in the hot spot problem. $\lambda = 0{\cdot}6928$, $Q(Z, V) = 2\exp(-Z^2)$ and $f(R, Z) = R\{1 - CZ^2\exp(-Z^2)\}$ where C takes the values 1, 2 and 5 for the curves —□—, —○— and —△— respectively.

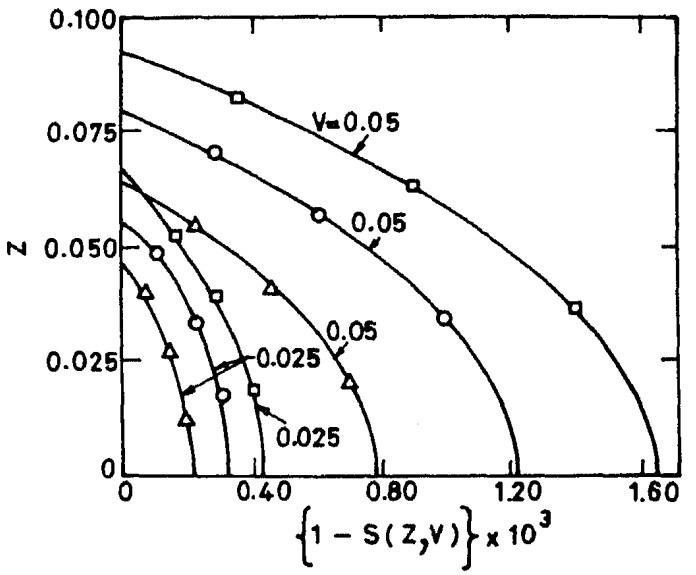

Figure 3. Effect of flux on the melting front in the hot spot problem. $\lambda = 0{\cdot}6928$, $f(R, Z) = R\{1 - 5Z^2\exp(-Z^2)\}$ and $Q(Z, V) = D\exp(-Z^2)$ where D takes the values 2, 2·5 and 3 for the curves —△—, —○— and —□— respectively.

temperature T_{II} does not effect the temperature T_I, the solution is valid for very short time (as a precaution). In figures 2 and 3, the melting front has been presented. In figure 2 a decrease in the initial temperature decreases more rapidly the melt spread along the surface than the penetration along the interior. In figure 3, an increase in prescribed flux increases more rapidly the melt penetration along the interior than the melt spread along the surface. The maximum values of the ratio $B(V)/|1 - S(0, V)|$ in figures 2 and 3 are 471·8 and 218·9 respectively. $b_1 = 0{\cdot}6656$, 0·4707 and 0·2976; $b_3 = -0{\cdot}2388$, $-0{\cdot}1531$ and $-0{\cdot}2252$ for the three initial temperatures in the same order as

$$16\sqrt{\pi\lambda}A_2(Z) = -4\sqrt{\pi}Q_1 - 20\sqrt{\pi}A_1^2 + 2\sqrt{\pi}\,\mathrm{erfc}\,(A_1)\frac{\partial f_1}{\partial R}\bigg|_{R=1}$$
$$-[8A_1\exp(-A_1^2) - \sqrt{\pi}\,\mathrm{erfc}\,A_1]f_1(1,Z)$$
$$+8A_1\exp(-A_1^2)\{f_2(1,Z)+f_3(1,Z)\}$$
$$+2\sqrt{\pi}(1+\mathrm{erf}\,A_1)\left(\frac{\partial f_2}{\partial R}+\frac{\partial f_3}{\partial R}\right)\bigg|_{R=1}, \tag{A1}$$

$$2\left(\frac{\partial f_2}{\partial R}+\frac{\partial f_1}{\partial R}\right)\bigg|_{R=1} + f_1(1,Z) = 4Q_1, \tag{A2}$$

$$2\{\exp(-A_1^2)+\sqrt{\pi}A_1(1+\mathrm{erf}\,A_1)\}\left(\frac{\partial f_2}{\partial R}+\frac{\partial f_3}{\partial R}\right)\bigg|_{R=1}$$
$$= 6\sqrt{\pi}A_1 + 2\sqrt{\pi}\,i\,\mathrm{erfc}\,(A_1)\frac{\partial f_1}{\partial R}\bigg|_{R=1} + 2(1+2A_2)\exp(-A_1^2)$$
$$\times\{f_1(1,Z)-f_2(1,Z)-f_3(1,Z)\} + \{\exp(-A_1^2)$$
$$-3\sqrt{\pi}A_1(1+\mathrm{erf}\,A_1)\}\{f_2(1,Z)+f_3(1,Z)\}$$
$$-4\sqrt{\pi}A_1\,\mathrm{erfc}\,(A_1)f_1(1,Z), \tag{A3}$$

$$96\sqrt{\pi\lambda}A_3(Z) = -16\sqrt{\pi}Q_2 - 240\sqrt{\pi\lambda}A_1A_2 - 30\sqrt{\pi\lambda}A_1^3$$
$$+\{-16A_1'^2 + 32A_1A_3 + 8A_2 + 16A_2^2 + 64A_1^2A_2$$
$$-32A_1^2A_2^2\}\exp(-A_1^2)\{f_1(1,Z)-f_2(1,Z)-f_3(1,Z)\}$$
$$+8\{\sqrt{\pi}A_1\,\mathrm{erfc}\,A_1 + \exp(-A_1^2)(3-6A_2-2A_1^2)\}f_1(1,Z)$$
$$+4\{7\sqrt{\pi}A_1\,\mathrm{erfc}\,A_1 - (3+4A_2)\exp(-A_1^2)\}\frac{\partial f_1}{\partial R}\bigg|_{R=1}$$
$$-8\sqrt{\pi}\,i\,\mathrm{erfc}\,(A_1)\frac{\partial^2 f_1}{\partial R^2}\bigg|_{R=1} + \exp(-A_1^2)(20A_2+16A_1^2-3)$$
$$\times\{f_2(1,Z)+f_3(1,Z)\} + 4\{(1+4A_2)\exp(-A_1^2)$$
$$+5\sqrt{\pi}A_1(1+\mathrm{erf}\,A_1)\}\left(\frac{\partial f_2}{\partial R}+\frac{\partial f_3}{\partial R}\right)\bigg|_{R=1}$$
$$+8\{\exp(-A_1^2)+\sqrt{\pi}A_1(1+\mathrm{erf}\,A_1)\}$$
$$\times\left(\frac{\partial^2 f_2}{\partial R^2}+\frac{\partial^2 f_3}{\partial R^2}\right)\bigg|_{R=1} - 4\exp(-A_1^2)\frac{\partial^2 f_1}{\partial Z^2}\bigg|_{R=1}$$

$$+4\exp(-A_1^2)\left(\frac{\partial^2 f_2}{\partial Z^2}+\frac{\partial^2 f_3}{\partial Z^2}\right)\Bigg|_{R=1}, \tag{A4}$$

$$A_1'=\frac{dA_1}{dZ},$$

$$2\frac{\partial^2 f_2}{\partial R^2}\Bigg|_{R=1}=-\frac{\partial f_2}{\partial R}\Bigg|_{R=1}+2\frac{\partial^2 f_1}{\partial R^2}\Bigg|_{R=1}+3\frac{\partial f_1}{\partial R}\Bigg|_{R=1}+4\sqrt{\pi}Q_2, \tag{A5}$$

$$\{2A_1\exp(-A_1^2)+\sqrt{\pi}(1+2A_1^2)(1+\operatorname{erf}A_1)\}\left(\frac{\partial^2 f_2}{\partial R^2}+\frac{\partial^2 f_3}{\partial R^2}\right)\Bigg|_{R=1}$$

$$=\{2A_1\exp(-A_1^2)-\sqrt{\pi}(1+2A_1^2)\operatorname{erfc}A_1\}\frac{\partial^2 f_1}{\partial R^2}\Bigg|_{R=1}$$

$$+2A_3\exp(-A_1^2)\{f_1(1,Z)-f_2(1,Z)-f_3(1,Z)\}$$

$$+2\{4A_1\exp(-A_1^2)-\sqrt{\pi}(2A_2+4A_1^2+1)\operatorname{erfc}A_1\}\frac{\partial f_1}{\partial R}\Bigg|_{R=1}$$

$$+3\sqrt{\pi}A_1^2+12\sqrt{\pi}A_2+\{4A_1\exp(A_1^2)$$

$$-\sqrt{\pi}(8A_2+4A_1^2+1/4)\operatorname{erfc}A_1\}f_1(1,Z)$$

$$-\{6A_1\exp(-A_1^2)+\sqrt{\pi}(1+4A_2+6A_1^2)(1+\operatorname{erf}A_1)\}$$

$$\times\left(\frac{\partial f_2}{\partial R}+\frac{\partial f_3}{\partial R}\right)\Bigg|_{R=1}-3\{A_1\exp(-A_1^2)+\sqrt{\pi}(4A_2+A_1^2)(1+\operatorname{erf}A_1)\}$$

$$\times\{[f_2(1,Z)+f_3(1,Z)]/2\}+8A_1A_2(1-A_2)\exp(-A_1^2)\{f_1(1,Z)$$

$$-f_2(1,Z)-f_3(1,Z)\}-\sqrt{\pi}\operatorname{erfc}(A_1)\frac{\partial^2 f_1}{\partial Z^2}\Bigg|_{R=1}$$

$$-\sqrt{\pi}(1+\operatorname{erf}A_1)\left(\frac{\partial^2 f_2}{\partial Z^2}+\frac{\partial^3 f_3}{\partial Z^2}\right)\Bigg|_{R=1}. \tag{A6}$$

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Rheology of polarizable non-piezoelectromagnetic material in relativity

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Abstract. Following the work of Carter on nonlinear perfectly elastic solid and perfect nonlinearly polarizable nonconducting solid, we have constructed models whose free gravitational field is of Petrov type *D*: (i) in inertial reference frame (IRF), (ii) with pure expansion and (iii) with pure rotation with the assumption that the flow field is expressible in terms of two real null vectors of the Newman–Penrose (N–P) tetrad. By using the strain variation equation, the necessary and sufficient conditions on the dynamical variables are obtained in Newman–Penrose version. We observe that the initial pressure tensor depends on the polarizable and electromagnetic properties of the material. Further, we conclude that there does not exist such a material with pure expansion but there exists such a material moving rigidly with or without rotation. We obtain the Hawking energy conditions and invariants for this material in IRF.

Keywords. Relativistic rheology; continuum mechanics; polarizable material; Newman–Penrose technique.

1. Introduction

Rheology as an ubiquitous discipline is well known [9, 12, 8]. In high speed mechanics, the concept of a rigid body is untenable [11]. Hence the concept of born-rigid medium, an isotropic solid and a perfectly elastic solid have been developed by Carter and Quintana [3], in the context of the relativistic continuum mechanics. For a covariant material tensor field, like deformation tensor field and vorticity tensor field, the convective derivative with respect to flow vector is the same as the Lie derivative but for a contravariant vector field, the connective derivative coincides with the orthogonal projection of the Lie derivative [7].

The word rheometrodynamics first appeared in Carter's [2] work as a generalization of Wheeler's [14] geometrodynamics. The purpose of our present work is to construct an exact model for a polarizable non-piezoelectromagnetic material (PNPEM) with the help of the energy momentum tensor by Carter [2] under the following assumptions:

(i) (a) The particles move rectilinearly with constant velocity (IRF), (b) purely expanding PNPEM and (c) purely rotating PNPEM.
(ii) The flow field is expressible in terms of the two real null vectors of the NP tetrad.
(iii) The free gravitational field is of Petrov type *D*.
(iv) The pressure tensor is divergence-free. (1)

investigations centre on fluids in a state of overall volume expansion or contraction (i.e. $\theta \neq 0$).

Section 2 of this paper contains the strain variation equation in terms of the kinematical variables (rotation, deformation) and the dynamical variables (density, divergence of the pressure tensor and gravitational tidal force). The necessary and sufficient initial conditions on the dynamical variables, consistent with the N–P equations, Bianchi identities and Maxwell equations are obtained for three different cases namely (i) IRF, (ii) purely expanding and (iii) purely rotating. Further, we have obtained the values of $E_c P^c$, $B_c M^c$ and initial pressure tensor. The PNPEM material with pure expansion does not exist, but in other two cases such a material does exist. In § 3, the Maxwell equations for PNPEM in IRF have been enumerated. In § 4 the three Hawking energy conditions [10] have been calculated. In the Appendix we list the nomenclature, IRF, N–P equivalent of IRF, the Ricci scalars, N–P equations, Bianchi identities, commutator relations, metric equations and the invariants.

2. Strain variation equation

In general relativistic rheology, Carter and Quintana's [4] identity (duly adjusted for change in signature) for the variation of the strain θ_{ab} is

$$\mathscr{L}_u \theta_{ab} = \underset{1}{L} + \underset{2}{L} + \underset{3}{L} + \underset{4}{L} + \underset{5}{L} + \underset{6}{L}, \tag{2}$$

where

$$\underset{1}{L} = h_{k(a}h_{b)}a(\rho^*_{;d} f^k - \rho^* f^k_{;d})(\rho^*)^{-2},$$

$$\underset{2}{L} = -C_{ab},$$

$$\underset{3}{L} = \frac{-4\pi G}{3}\rho^* h_{ab},$$

$$\underset{4}{L} = w_{ka}w^k_b + 2w^k_{(a}\theta_{b)} + g^{kd}\theta_{ak}\theta_{bd},$$

$$\underset{5}{L} = C^{-2}\rho^{*-2}(h_{ak}h_{bd}f^k f^d + \rho^* p^{kd}\theta_{kd}\theta_{ab}),$$

$$\underset{6}{L} = -4\pi C^{-2}(P_{ab} - \tfrac{2}{3}p^k_k h_{ab}).$$

Here

$\underset{1}{L}, \underset{2}{L}, \underset{3}{L}$ and $\underset{4}{L}$ correspond to the Newtonian rheology ($C \to \infty$),

$\underset{1}{L}, \underset{4}{L}$ and $\underset{5}{L}$ correspond to the special relativistic rheology ($G = 0 = C_{ab}$), and

$\underset{6}{L}$ is the general relativistic correction term.

For non-piezoelectromagnetic material where the polarization depends upon the electromagnetic field linearly (for a very artificial special case) the energy momentum tensor for the material is given by Carter [2]:

$$T_{ab}|_M = \underset{0}{\rho} u_a u_b - \tfrac{1}{2}(E_c P^c - B^c M_c) h_{ab} + \tfrac{1}{2}(E_c P^c - B^c M_c) u_a u_b$$

$$+ \underset{0}{p}_{ab} + E_{(a}P_{b)} + B_{(a}M_{b)}, \tag{3}$$

where $\underset{0}{\rho}$ and $\underset{0}{p}_{ab}$ are the initial density and pressure tensor of the material respectively. The density and pressure tensor are given by

$$\rho^* = \underset{0}{\rho} + K$$

and

$$P_{ab} = \underset{0}{p}_{ab} - K h_{ab} + E_{(a}P_{b)} + B_{(a}M_{b)},$$

where

$$K = \tfrac{1}{2}(E_c P^c - B_c M^c).$$

2.1. *Null rheology*

Since $E^a, B^a, \ldots$ are the flow orthogonals, a consistent choice of the vector fields can be enumerated as

$$u^a = (2^{-1/2}, 2^{-1/2}, 0, 0),$$

$$E^a = (E, -E, 0, 0),$$

$$B^a = (B, -B, 0, 0),$$

$$H^a = (H, -H, 0, 0),$$

$$M^a = (M, -M, 0, 0),$$

$$P^a = (P, -P, 0, 0)$$

and

$$D^a = (D, -D, 0, 0)$$

where $E = -E^c n_c = E^c \mathbf{l}_c$ etc.

We assume that the tetrad is parallely propagated along the congruences $\mathbf{l}^a$ and n^a, hence $k = \pi = \varepsilon = 0$ and $\tau = \nu = \gamma = 0$.

2.2. *T_{ab} in IRF*

We have Einstein's field equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} = -T_{ab}. \tag{4}$$

From the Ricci identity

$$u_{a;bc} - u_{a;cb} = u_k R^k_{abc} \tag{5}$$

$$u_k R^k_{abc} = 0 \tag{6}$$

and hence

$$u_k R^k_b = 0. \tag{7}$$

Contracting the decomposition of the Riemann tensor

$$R_{abcd} = C_{abcd} + g_{a[c}R_{b]d} + g_{b[d}R_{c]a} + \tfrac{1}{3}g_{a[d}g_{c]b}R$$

with $u^b u^d$ and using (6), (7) and (4) we get the relation

$$T_{ab}|_{\mathrm{IRF}} = 2C_{ab} + \tfrac{1}{3}Ru_a u_b + \tfrac{1}{6}Rg_{ab}, \tag{8}$$

where

$$C^{[ab]} = 0, \quad C^{ab}u_b = 0, \quad C^a_a = 0. \tag{9a–c}$$

2.2a *Time and spatial components.* Covariantly differentiating (8), using (9b) and then contracting with u_a, we get

$$R_{,b}u^b = 0 \quad \text{or} \quad \dot{R} = 0. \tag{10}$$

This is the equation of continuity. Using (9b) and the u-orthogonal character of some vectors

$$u^b T_{ab}|_{\mathrm{IRF}} = u^b T_{ab}|_M$$

i.e. equality of two time components give

$$\underset{0}{\rho} + K = R/2$$

or

$$\rho^* = R/2. \tag{11a}$$

Similarly

$$h^{ab}T_{ab}|_{\mathrm{IRF}} = h^{ab}T_{ab}|_M$$

i.e. equality of spatial components give

$$\underset{0}{p^a_a} - \frac{R}{2} - \tfrac{1}{2}E_a P^a + \tfrac{5}{2}B^a M_a = 0. \tag{11b}$$

From (11a) and (11b) we get

$$\underset{0}{\rho} - \underset{0}{p^a_a} + E_a P^a - 3B^a M_a = 0. \tag{11c}$$

This does not necessarily imply $\underset{0}{\rho} < 0$, so it is physically tenable. The values of $E_a P^a$ and $B_a M^a$ depend on three scalars viz the initial density, the trace of the initial pressure and the Ricci scalar and are given by

$$B^a M_a = -\tfrac{1}{2}(-R + \underset{0}{\rho} + \underset{0}{p^a_a}). \tag{12b}$$

The strain variation equation (2) for IRF with the conditions (1) becomes

$$C_{ab} = -\tfrac{1}{6}\rho^* h_{ab} - \tfrac{1}{2}(P_{ab} - \tfrac{2}{3}P^k_k h_{ab}). \tag{13}$$

Substituting the values of $E_c P^c$ and $B^c M_c$ from (12a, b) the identity $C_{ab}g^{ab} = 0$ implies $\rho^* = R/2$ which is the same as (11a). We assume that

$$\left.\begin{aligned} &\underset{0}{p_{12}} = \underset{0}{p_{cd}}\mathbf{l}^c n_d = \underset{0}{p_{cd}} n^c \mathbf{l}^d = -\underset{0}{p_{cd}}\mathbf{l}^c \mathbf{l}^d = -\underset{0}{p_{cd}} n^c n^d \\ \text{and} \\ &\underset{0}{p_{34}} = \underset{0}{p_{cd}} m^c \bar{m}^d = \underset{0}{p_{cd}} \bar{m}^c m^d \end{aligned}\right\} \tag{14}$$

and the other coefficients vanish. Equation (13) in the N–P formalism with Petrov type D gravitational field takes the form

$$\begin{aligned} &\tfrac{1}{2}(\psi_2 + \bar{\psi}_2)(\mathbf{l}_a\mathbf{l}_b - \mathbf{l}_a n_b - n_a\mathbf{l}_b + n_a n_b - m_a\bar{m}_b - \bar{m}_a m_b) \\ &= -\tfrac{1}{2}(\tfrac{1}{3}\underset{0}{\rho} - \tfrac{4}{3}\underset{0}{p_{12}} + \tfrac{1}{3}\underset{0}{p_{34}} + \tfrac{8}{3}BM)(\mathbf{l}_a\mathbf{l}_b - \mathbf{l}_a n_b - n_a\mathbf{l}_b \\ &\quad + n_a n_b - m_a\bar{m}_b - \bar{m}_a m_b) \end{aligned}$$

i.e.

$$\tfrac{4}{3}\underset{0}{p_{12}} - \tfrac{1}{3}\underset{0}{p_{34}} = \psi_2 + \bar{\psi}_2 + \tfrac{1}{3}\underset{0}{\rho} + \tfrac{8}{3}BM. \tag{15}$$

Thus the initial pressure needed for a solid star filled with non-piezoelectromagnetic material to have divergence-free pressure tensor and a Petrov type D free gravitational field is determined by (15).

2.3. *Purely expanding PNPEM ($\theta \neq 0$), θ IRF*

Here we relax the condition $\theta = 0$ of IRF i.e. only the kinematical constant θ non-zero and follow the same procedure to get the following results.

$$u_{a;b} = \tfrac{1}{3}\theta h_{ab}, \tag{16}$$

$$T_{ab}|_{\theta\text{IRF}} = 2C_{ab} + \tfrac{1}{3}Ru_a u_b + \tfrac{1}{6}Rg_{ab} - \tfrac{1}{9}\theta^2 g_{ab} + \tfrac{4}{9}\theta u_a u_b. \tag{17}$$

The equation of continuity is

$$\tfrac{1}{3}\theta^3 + \tfrac{3}{8}\dot{R} + \tfrac{1}{4}R\theta = 0. \tag{18}$$

The time and spatial components are respectively

$$\rho^* = \tfrac{1}{3}\theta^2 + \tfrac{1}{2}R; \quad \dot{\theta} = 0 \tag{19a, b}$$

and

$$\underset{0}{p^a_a} - \frac{R}{2} + \tfrac{1}{3}\theta^2 - \tfrac{1}{2}E_a P^a + \tfrac{5}{2}B^a M_a = 0. \tag{19c}$$

$$B^c M_c = \tfrac{1}{2}R - \tfrac{1}{2}\underset{0}{\rho} - \tfrac{1}{2}\underset{0}{p^a_a}. \tag{20b}$$

The contraction of the strain variation equation with g^{ab} and using

$$\mathscr{L}_u\theta_{ab} = \mathscr{L}_u(\tfrac{1}{3}\theta h_{ab}) = \tfrac{1}{9}\theta^2 h_{ab} \quad \text{since } \dot{\theta} = 0$$

gives

$$R\theta^2 = 0. \tag{21}$$

This implies that either $R = 0$ or θ must be zero. But as we have taken θ to be non-zero we must have $R = 0$. But the equation of continuity with $R = 0$ gives $\theta = 0$, which contradicts our assumption. Hence we conclude that the purely expanding PNPEM will not exist, consistent with the conditions (1).

2.4. *Purely rotating PNPEM ($w_{ab} \neq 0$) WIRF*

All the kinematical variables except w_{ab} are zero

$$w_{ab} = \varepsilon_{abcd} w^c u^d, \tag{22}$$

$$w_{ab}w^{ab} = w_a w^a = -w^2 \tag{23}$$

and

$$w_{a;b} = H_{ab} + u_a u_b w^c_{;c} - w_a \dot{u}_b, \tag{24}$$

where w^a is the rotation vector and

$$H_{ab} = \tfrac{1}{2}\varepsilon_{arpq} C^{pq}_{bs} u^r u^s$$

In (5), using $u_{a;b} = w_{ab}$, we get

$$\begin{aligned} u_k R^k_{abc} = {} & \varepsilon_{abed} H^e_c u^d - \varepsilon_{aced} H^e_b u^d - \varepsilon_{abed} w^e u_c u^d \\ & - \varepsilon_{aced} w^e u_b u^d + w_c w_a u_b - w_a w_b u_c - 2g_{ca} u_b w^2 \\ & + 2g_{ab} u_c w^2. \end{aligned} \tag{25}$$

$$u_k R^k_b = -4w^2 u_b. \tag{26}$$

The energy momentum tensor is

$$\begin{aligned} T_{ab}|_{\text{WIRF}} = {} & 2C_{ab} + \tfrac{1}{3}R u_a u_b + \tfrac{1}{6}R g_{ab} + 4w^2 u_a u_b - 2w_a w_b \\ & + 2\varepsilon_{baed} w^e u^d. \end{aligned} \tag{27}$$

The equation of continuity is

$$(1/24\dot{R}) + w\dot{w} = 0. \tag{28}$$

The time and spatial components are respectively

and

$$\underset{0}{p^a_a} - \frac{R}{2} + 4w^2 - \tfrac{1}{2}E_a p^a + \tfrac{5}{2}B^a M_a = 0. \tag{29b}$$

$$E_a P^a = -\tfrac{5}{2}\underset{0}{\rho} - \tfrac{1}{2}\underset{0}{p^a_a} + \tfrac{3}{2}R + 8w^2 \tag{30a}$$

and

$$B_a M^a = -\tfrac{1}{2}(-R + \underset{0}{\rho} + \underset{0}{p^a_a}). \tag{30b}$$

The value of $B_a M^a$ is the same as that in IRF but the value of $E_a P^a$ differed by $8w^2$ as that in IRF. The contraction of strain variation (2) with g^{ab} gives

$$\rho^* = 4(\tfrac{1}{8}R + w^2)$$

i.e. the density of a purely rotating PNPEM depends linearly on R and w^2 and is same as (29a). Hence this case is physically tenable. The strain variation equation

$$C_{ab} = (-\tfrac{1}{6}\rho^* + 2w^2 + \tfrac{1}{3}\underset{0}{p^c_c} + \tfrac{1}{12}E_c P^c + \tfrac{7}{12}B^c M_c)h_{ab}$$

$$-\tfrac{1}{2}(\underset{0}{p_{ab}} + E_{(a}P_{b)} + B_{(a}M_{b)}) - w_a w_b. \tag{31}$$

In N–P formalism for the choice $w_a = (w/\sqrt{2})(\mathbf{l}_a - n_a)$, (31) is equivalent to

$$\tfrac{1}{2}(\psi_2 + \bar{\psi}_2)(\mathbf{l}_a\mathbf{l}_b - \mathbf{l}_a n_b - n_a\mathbf{l}_b + n_a n_b - m_a\bar{m}_b - \bar{m}_a m_b)$$

$$= (\tfrac{1}{2}\underset{0}{\rho} - \tfrac{1}{3}R - \tfrac{1}{2}\underset{0}{p_{34}} + \tfrac{11}{8}w^2)(\mathbf{l}_a\mathbf{l}_b + n_a n_b - \mathbf{l}_a n_b - n_a\mathbf{l}_b)$$

$$+ (\tfrac{1}{2}\underset{0}{\rho} - \tfrac{1}{3}R - \tfrac{1}{2}\underset{0}{p_{34}} - 2w^2)(m_a\bar{m}_b + \bar{m}_a m_b). \tag{32}$$

Hence comparing the coefficients we get

$$\underset{0}{p_{34}} = \underset{0}{\rho} - \tfrac{2}{3}R - \tfrac{5}{8}w^2 \tag{33a}$$

and

$$\psi_2 + \bar{\psi}_2 = \tfrac{27}{8}w^2. \tag{33b}$$

Thus the initial pressure tensor component $\underset{0}{p_{34}}$ and the $\mathrm{Re}(\psi_2)$ are determined in terms of initial density Ricci scalar and rotation scalar and hence the pressure tensor is determined in terms of only the Ricci and rotation scalars as

$$P_{ab} = (\tfrac{1}{12}R - \tfrac{19}{8}w^2)(\mathbf{l}_a\mathbf{l}_b + n_a n_b - \mathbf{l}_a n_b - n_a\mathbf{l}_b)$$

$$+ (\tfrac{-7}{6}R - \tfrac{37}{8}w^2)(m_a\bar{m}_b + \bar{m}_a m_b). \tag{34}$$

The values of $E_c P^c$ and $B_c M^c$ in N–P formalism are

$$-\tfrac{1}{2}E_c P^c = EP = \tfrac{3}{4}\underset{0}{\rho} + \tfrac{1}{16}w^2 - \tfrac{5}{12}R + \tfrac{1}{2}\underset{0}{p_{12}}, \tag{35a}$$

For the choice

$$w_a = (w/\sqrt{2})(m_a + \bar{m}_a),$$

we get

$$\tfrac{1}{2}(\psi_2 + \bar{\psi}_2)(\mathbf{l}_a\mathbf{l}_b - \mathbf{l}_a n_b - n_a\mathbf{l}_b + n_a n_b - m_a\bar{m}_b - \bar{m}_a m_b)$$

$$= (\tfrac{1}{2}\underset{0}{\rho} - \tfrac{1}{3}R - \tfrac{1}{2}\underset{0}{p}_{34} + \tfrac{7}{8}w^2)(\mathbf{l}_a\mathbf{l}_b - \mathbf{l}_a n_b - n_a\mathbf{l}_b + n_a n_b)$$

$$+ (\tfrac{1}{2}\underset{0}{\rho} - \tfrac{1}{3}R - \tfrac{1}{2}\underset{0}{p}_{34} - \tfrac{5}{2}w^2)(m_a\bar{m}_b + \bar{m}_a m_b) - \frac{w^2}{2}(m_a m_b + \bar{m}_a\bar{m}_b). \tag{36}$$

Comparing the coefficients of both sides, we obtain $w = 0$, which is a contradiction. Hence the choice $w_a = (w/\sqrt{2})(m_a + \bar{m}_a)$ is physically untenable.

For all the above three cases the vorticity equation $\mathscr{L}_u w_{ab}$ is identically zero, and hence the rotation tensor is physically constant for all the three cases.

3. Maxwell equations

The Maxwell equations can be written in the standard form [4]

$$[B^a]^{\cdot} + \theta B^a = -\varepsilon^{abc}(\dot{u}_b + \nabla_b)E_c,$$

$$h^b_c\nabla_b B^c = 2w^b E_b$$

and

$$[D^a]^{\cdot} + \theta D^a = \varepsilon^{abc}(\dot{u}_b + \nabla_b)H_c - h^a_b J^b,$$

$$h^b_c\nabla_b D^c = -2w^b H_b - u_b J^b.$$

$[\ \ ]^{\cdot}$ is the convected derivative and $\dot{B}_a$ denotes the covariant differentiation w.r.t. proper time.

$$[B^a]^{\cdot} = \dot{B}^a - B^e(\dot{u}_e + \nabla_e)u^a, \quad \nabla_e u^a = u^a_{;e}.$$

For IRF these equations take the form

$$B^a_{;a} = 0; \tag{37a}$$

$$\dot{B}_b - B^a u_{b;a} + \varepsilon^{ca}_b H_{a;c} = 0; \tag{37b}$$

$$D^a_{;a} = +u_a J^a; \tag{37c}$$

$$\varepsilon^{ca}_b H_{a;c} = \dot{D}_b - D^a u_{b;a} + h_{ba}J^a. \tag{37d}$$

3.1. *N–P concomitant of Maxwell equations*

The N–P concomitant of Maxwell equations for IRF are

where

$$V = \varepsilon_{abcd} l^a n^b m^c \bar{m}^d$$

is the infinitesimal volume.

3.2. *Stress tensor*

$$T^{ab} = (\tfrac{1}{2}\underset{0}{\rho} + 2BM - \underset{0}{p}_{12})(l^a l^b + n^a n^b) + (\tfrac{1}{2}\underset{0}{\rho} - EP - BM + \underset{0}{p}_{12}) \times (l^a n^b + n^a l^b) + (-EP + BM + \underset{0}{p}_{34})(m^a \bar{m}^b + \bar{m}^a m^b), \tag{39}$$

$$T = \underset{0}{\rho} - 4BM + 2\underset{0}{p}_{12} - 2\underset{0}{p}_{34}, \tag{40}$$

$$-R^{ab} = T^{ab} - \tfrac{1}{2} T g^{ab} = (\tfrac{1}{2}\underset{0}{\rho} + 2BM - \underset{0}{p}_{12})(l^a l^b + n^a n^b) + (-EP + BM + \underset{0}{p}_{34}) \times (l^a n^b + n^a l^b) + (\tfrac{1}{2}\underset{0}{\rho} - EP + \underset{0}{p}_{12} - BM)(m^a \bar{m}^b + \bar{m}^a m^b). \tag{41}$$

4. Hawking energy conditions

These energy conditions are expressed in terms of time-like and null congruences. The three inequalities which are necessary for a stress energy momentum tensor T_{ab} to be physically reasonable are known as energy conditions [10]: The characterization of the three energy conditions can be described as

(i) The weak energy condition states that

$$T_{ab} u^a u^b \geqslant 0, \quad T_{ab} n^a n^b \geqslant 0. \tag{42a}$$

(ii) The dominant energy condition:

$$T^{ab} u_a u_b \geqslant 0. \tag{42b}$$

From this, we infer that the energy flux congruence $T^{ab} u_a$ is either time-like or null. This condition also states that the velocity of the matter is less than or at the most equal to the velocity of light.

(iii) The strong energy condition: It states that

$$T^{ab} u_a u_b \geqslant u_a u^a \tfrac{1}{2}(T). \tag{43}$$

$$-\tfrac{1}{2}(E_kP^k - B^kM_k)h^{ab} + \tfrac{1}{2}E^aP^b$$

$$+\tfrac{1}{2}E^bP^a + \tfrac{1}{2}B^bM^a + \tfrac{1}{2}B^aM^b.$$

$$T^{ab}u_au_b = \underset{0}{\rho} + \tfrac{1}{2}(E_kP^k - B^kM_k).$$

(i) *Weak energy condition*
From (43),

$$T^{ab}u_au_b = \underset{0}{\rho} + \tfrac{1}{2}E_aP^a - \tfrac{1}{2}B_aM^a \geqslant 0 \quad \text{or} \quad \rho^* \geqslant 0.$$

i.e. the density of the material must be greater than zero and

$$T^{ab}n_an_b = \underset{0}{p^{ab}}n_an_b + \tfrac{1}{2}E^aP^bn_an_b + \tfrac{1}{2}E^bP^an_an_b$$

$$+\tfrac{1}{2}B^aM^bn_an_b + \tfrac{1}{2}E^bM^an_an_b \geqslant 0,$$

where n^a is a null vector.

For IRF these reduce to

$$\underset{0}{\rho} \geqslant 4BM - 2\underset{0}{p_{12}} + 2\underset{0}{p_{34}}$$

and

$$\underset{0}{\rho} \geqslant 4BM - 2\underset{0}{p_{12}}$$

for n_a to be a null vector from a tetrad $(\mathbf{l}_a, n_a, m_a, \bar{m}_a)$.

(ii) *Dominant energy condition*

$$T^{ab}u_au_b \geqslant 0 \quad \text{i.e. } \rho^* \geqslant 0.$$

i.e. same as (44a). For IRF this condition is the same as (45a).

(iii) *Strong energy condition*
From (43)

$$T^{ab}u_au_b \geqslant \tfrac{1}{2}T \text{ since } u^au_a = +1,$$

$$T = T^a_a = \underset{0}{\rho} + \underset{0}{p^a_a} + 2B^kM_k.$$

Hence for strong energy condition, we must have

$$\underset{0}{\rho} + \tfrac{1}{2}E_kP^k - \tfrac{1}{2}B^kM_k \geqslant \tfrac{1}{2}\underset{0}{\rho} + \tfrac{1}{2}\underset{0}{p^a_a} + B^kM_k.$$

i.e.

$$\tfrac{1}{2}(\underset{0}{\rho} + E_aP^a - 3B^aM_a) \geqslant \tfrac{1}{2}\underset{0}{p^a_a}$$

or

or IRF this condition reduces to

$$\underset{0}{p_a^a} \leqslant 2\underset{0}{p_{12}} - 2\underset{0}{p_{34}}. \tag{47}$$

$$\underset{0}{p_a^a} = 2\underset{0}{p_{12}} - 2\underset{0}{p_{34}}.$$

ıce this condition is satisfied.

Appendix

Nomenclature

density of the material
unit time-like flow vector field
electric field vector
magnetic induction vector
electric polarization density vector
magnetic polarization density vector
$= [(4\pi)^{-1}E^a + P^a]$, the electric induction vector
$= [(4\pi)^{-1}B^a - M^a]$, the magnetic field vector
deformation tensor
$= (g_{ab} - u_a u_b)$, the 3-dimensional projection tensor
$= (\rho^* u^a u^b + p^{ab})$, the energy momentum tensor
$= (p^{ab}_{;b})$, the pressure flux vector
$(-u^c u^d C_{acbd})$, the gravitational tidal force
$= \varepsilon^{abc} B_c$
$= (2u^a E^b + B^{ab})$, the Maxwell field tensor
$= \varepsilon^{abc} M_c$
$= (2u^a p^b - M^{ab})$, the polarization tensor
$= (F^{ab} + 4\pi P^{ab})$
$= (2u^a D^b + \varepsilon^{abc} H_c)$, the antisymmetric electromagnetic displacement tensor
$= \frac{1}{2}\varepsilon^{abc} u_{a;b}$
$= \varepsilon^{abcd} u_d$
real null vectors
conjugate complex null vectors
$= -C_{abcd}\bar{m}^a n^b l^c m^d$, tetrad component of the Weyl tensor

$\sigma, \tau, \nu, \mu,$
$\pi, \alpha, \beta, \gamma, \varepsilon,$ the spin coefficients in NP formalism [5].

IRF

oud of test particles which moves rigidly and without rotation represents an inertial rence frame [1]. Here the 4-velocity field $u^a(x)$ of the particles ($u^a u_a = +1$) has the

$$\sigma_{ab} = u_{(a;b)} - \dot{u}_{(a}u_{b)} - \tfrac{1}{3}\theta h_{ab} = 0 \quad \text{(vanishing shear)}, \tag{A3}$$

$$w_{ab} = u_{[a;b]} - \dot{u}_{[a}u_{b]} = 0 \quad \text{(vanishing rotation)}. \tag{A4}$$

Accordingly the identity

$$u_{a;b} = w_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} + \dot{u}_a u_b$$

for an IRF implies

$$u_{a;b} = 0 \quad \text{(covariantly constant flow)}. \tag{A5}$$

A.3. *Newman–Penrose equivalent of IRF*

(i) $u^a = 2^{-1/2}(\mathbf{l}^a + n^a)$.

(ii) $\dot{u}^a = 0$ iff

$$\gamma + \bar{\gamma} + \varepsilon + \bar{\varepsilon} = \nu + \pi - \bar{k} - \bar{\tau} = 0.$$

(iii) $u_{[a;b]} = 0$ iff

$$\bar{\tau} - \nu - (\alpha + \bar{\beta}) = \varepsilon + \bar{\varepsilon} + \gamma + \bar{\gamma} = \bar{\rho} - \rho + \bar{\mu} - \mu = \pi - \bar{k} - (\alpha + \bar{\beta}) = 0.$$

(iv) $u_{(a;b)} = 0$ iff

$$\gamma + \bar{\gamma} = \varepsilon + \bar{\varepsilon} = -\bar{\tau} + \nu = \bar{\sigma} - \lambda = \bar{\rho} - \mu = \alpha + \bar{\beta} = 0.$$

(v) The tetrad is parallely propagated along $\mathbf{l}^a$ and n^a i.e., $k = \pi = \varepsilon = 0$ and $\tau = \nu = \gamma = 0$ respectively. Therefore, in all, $\bar{\sigma} - \lambda = \bar{\rho} - \mu = \alpha + \bar{\beta} = 0$,

$$k = \pi = \varepsilon = 0, \quad \tau = \nu = \gamma = 0. \tag{A6}$$

A.4. *Ricci scalars*

In N–P formalism, the nonvanishing tetrad components of R_{ab} and R are given by

$$\phi_{00} = -\tfrac{1}{2}R_{ab}\mathbf{l}^a\mathbf{l}^b = \tfrac{1}{2}(\underset{0}{\tfrac{1}{2}\rho} + 2BM - \underset{0}{p_{12}}), \tag{A7}$$

$$\phi_{11} = -\tfrac{1}{4}R_{ab}(\mathbf{l}^a n^b + m^a \bar{m}^b)$$

$$= \tfrac{1}{24}R = \tfrac{1}{24}(\underset{0}{\rho} - 4BM + 2\underset{0}{p}_{12} - 2\underset{0}{p}_{34}). \tag{A10}$$

A.5. *N–P equations*

There are only six non-vanishing independent N–P equations which are given by

(i) $$D\rho = -\Delta\rho = \rho^2 + \tfrac{1}{4}\underset{0}{\rho} + BM - \underset{0}{p}_{12}. \tag{A11}$$

(ii) $$D\alpha = -\Delta\alpha = \rho\alpha. \tag{A12}$$

(iii) $$\psi_2 = \tfrac{1}{6}\underset{0}{\rho} + \tfrac{4}{3}BM - \tfrac{2}{3}\underset{0}{p}_{12} + \tfrac{1}{6}\underset{0}{p}_{34}. \tag{A13}$$

(iv) $$EP = 3BM + \tfrac{1}{2}\underset{0}{\rho} - \underset{0}{p}_{12} + \underset{0}{p}_{34}. \tag{A14}$$

(v) $$\delta\rho = 0. \tag{A15}$$

(vi) $$\delta\alpha - \bar{\delta}\bar{\alpha} = \rho\bar{\rho} + 4\alpha\bar{\alpha} - \tfrac{1}{4}\underset{0}{\rho} + \tfrac{3}{2}\underset{0}{p}_{12} - \tfrac{1}{2}\underset{0}{p}_{34} - 3BM. \tag{A16}$$

A.6. *Bianchi identities*

Only six Bianchi idéntities survive and these are

(i) $$\delta(\tfrac{1}{4}\underset{0}{\rho} + BM - \tfrac{1}{2}\underset{0}{p}_{12}) = 0 \quad \text{i.e. } \delta(\phi_{00}) = 0. \tag{A17}$$

(ii) $$\sigma\psi_2 = 0 \Rightarrow \sigma = 0 \quad \text{since } \psi_2 \neq 0. \tag{A18}$$

(iii) $$D(-\tfrac{3}{4}\underset{0}{\rho} - 7BM + \tfrac{7}{2}\underset{0}{p}_{12} - \underset{0}{p}_{34}) - \Delta(\tfrac{1}{4}\underset{0}{\rho} + BM - \tfrac{1}{2}\underset{0}{p}_{12})$$

$$= (\rho \blacklozenge \bar{\rho})(-\underset{0}{\rho} - 8BM + 4\underset{0}{p}_{12} - \underset{0}{p}_{34}). \tag{A19}$$

(iv) $$\Delta(\tfrac{3}{4}\underset{0}{\rho} + 7BM - \tfrac{7}{2}\underset{0}{p}_{12} + \underset{0}{p}_{34}) + D(\tfrac{1}{4}\underset{0}{\rho} + BM - \tfrac{1}{2}\underset{0}{p}_{12})$$

$$= -(\rho + \bar{\rho})(-\underset{0}{\rho} - 8BM + 4\underset{0}{p}_{12} - \underset{0}{p}_{34}). \tag{A20}$$

(A19) and (A20) will imply

$$(D + \Delta)\phi_{11} = 0.$$

(v) $$D(-2BM + \underset{0}{p}_{12} - \tfrac{1}{2}\underset{0}{p}_{34}) + \Delta(\tfrac{1}{4}\underset{0}{\rho} + BM - \tfrac{1}{2}\underset{0}{p}_{12})$$

$$= (\rho + \bar{\rho})(-\tfrac{1}{2}\underset{0}{\rho} - 4BM + 2\underset{0}{p}_{12} - \tfrac{1}{2}\underset{0}{p}_{34}). \tag{A21}$$

(vi) $$\Delta(-2BM + \underset{0}{p}_{12} - \tfrac{1}{2}\underset{0}{p}_{34}) + D(\tfrac{1}{4}\underset{0}{\rho} + BM + \tfrac{1}{2}\underset{0}{p}_{12})$$

$$(D + \Delta)\Lambda = 0.$$

Conclusion

From (A12) we find that the derivative of α along real congruences is proportional to α itself and from (A15), ρ is constant along m^a.

From Bianchi identities, we conclude that ϕ_{00} is constant along m^a and the shear of $\mathbf{l}^a$ congruence must be zero. ϕ_{11} is constant along $(\mathbf{l}^a + n^a)$ congruence.

A.7. *Commutator relations*

$$[\Delta, D]\phi = 0,$$

$$[\delta, D]\phi = -\bar{\rho}\delta\phi, \quad \text{since } \sigma = 0 \text{ from Bianchi identity,}$$

$$[\delta, \Delta]\phi = \bar{\rho}\delta\phi,$$

$$[\bar{\delta}, \delta]\phi = (\rho - \bar{\rho})(D - \Delta)\phi + 2\alpha\delta\phi - 2\bar{\alpha}\bar{\delta}\phi. \tag{A23}$$

A.8. *Metric equations*

$$\Delta l^a - Dn^a = 0,$$

$$\delta l^a - Dm^a = -\bar{\rho}m^a,$$

$$\delta n^a - \Delta m^a = \bar{\rho}m^a, \tag{A24}$$

$$\bar{\delta}m^a - \delta\bar{m}^a = (\rho - \bar{\rho})(\mathbf{l}^a - n^a) + 2\alpha m^a - 2\bar{\alpha}\bar{m}^a.$$

A.9. *Invariants*

The 4 invariants of T^{ab} for PNPEM material in IRF are evaluated and found as follows:

The stress energy-momentum tensor of PNPEM is given by

$$T^{ab} = (\tfrac{1}{2}\underset{0}{\rho} + 2BM - \underset{0}{p}_{12})(\mathbf{l}_a\mathbf{l}_b + n^a n^b - m^a\bar{m}^b - \bar{m}^a m^b)$$

$$+ (-4MB + 2\underset{0}{p}_{12} - \underset{0}{p}_{34})(\mathbf{l}^a n^b + n^a\mathbf{l}^b), \quad \text{(from A14)}$$

$$T^a_a = 2(\tfrac{1}{2}\underset{0}{\rho} - \underset{0}{p}_{34} + \underset{0}{p}_{12} - 2BM), \tag{A25}$$

$$T^b_a T^a_b = \underset{0}{\rho}^2 + (8BM - 4\underset{0}{p}_{12})\underset{0}{\rho} - 48\underset{0}{p}_{12}BM + 16\underset{0}{p}_{34}BM$$

$$+ 12\underset{0}{p}{}^2_{12} + 2\underset{0}{p}{}^2_{34} - 8\underset{0}{p}_{12}\underset{0}{p}_{34} + 48B^2M^2, \tag{A26}$$

$$T^c_a T^b_c T^a_b = \tfrac{1}{2}\underset{0}{\rho}^3 - 3\underset{0}{\rho}^2 BM + \tfrac{3}{2}\underset{0}{\rho}^2\underset{0}{p}_{12} - \tfrac{3}{2}\underset{0}{p}_{12}\underset{0}{\rho}^2 - 208B^3M^3$$

$$+ 312B^2M^2 \underset{0}{p_{12}} - 120B^2M^2 \underset{0}{p_{34}} - 156BM \underset{0}{p_{12}^2} + 26 \underset{0}{p_{12}^3}$$

$$- 30 \underset{0}{p_{12}^2} \underset{0}{p_{34}} - 36B^2M^2 \underset{0}{\rho} + 36BM \underset{0}{p_{12}} \underset{0}{\rho} - 12BM \underset{0}{\rho_{34}} \underset{0}{\rho}$$

$$+ 120BM \underset{0}{p_{12}} \underset{0}{p_{34}} - 9\rho \underset{0}{p_{12}} + 6 \underset{0}{p_{12}} \underset{0}{p_{34}} \rho - 24BM \underset{0}{p_{34}^2}$$

$$+ 12 \underset{0}{p_{12}} \underset{0}{p_{34}^2} - 2 \underset{0}{p_{34}^3}. \tag{A27}$$

$$T_a^d T_d^c T_c^b T_b^a = \tfrac{1}{8} \underset{0}{\rho^4} + 1344B^4M^4 + 84 \underset{0}{p_{12}^4} + 132B^2M^2 \underset{0}{\rho^2}$$

$$+ 240B^2M^2 \underset{0}{p_{34}^2} + 2016B^2M^2 \underset{0}{p_{12}^2} + 18 \underset{0}{\rho^2} \underset{0}{p_{12}^2}$$

$$+ 2 \underset{0}{p_{34}^4} + 50 \underset{0}{p_{12}^2} \underset{0}{p_{34}^2} + 4BM \underset{0}{\rho^3} + 24BM \underset{0}{p_{34}} \rho^2$$

$$- 64BM \underset{0}{p_{12}} \underset{0}{\rho^2} - 2 \underset{0}{p_{12}} \underset{0}{\rho^3} + 448B^3M^3 \underset{0}{\rho}$$

$$+ 896B^3M^3 \underset{0}{p_{34}} - 2688B^3M^3 \underset{0}{p_{12}} - 672B^2M^2 \underset{0}{p_{12}} \underset{0}{\rho}$$

$$- 968B^2M^2 \underset{0}{p_{12}} \underset{0}{p_{34}} + 3 \underset{0}{p_{34}^2} \underset{0}{\rho^2} - 12 \underset{0}{p_{12}} \underset{0}{p_{34}} \underset{0}{\rho^2}$$

$$+ 336BM \underset{0}{p_{12}^2} \underset{0}{\rho} + 672BM \underset{0}{p_{12}^2} \underset{0}{p_{34}} - 672BM \underset{0}{p_{12}^3}$$

$$- 56 \underset{0}{p_{12}^3} \underset{0}{\rho} - 112 \underset{0}{p_{12}^3} \underset{0}{p_{34}} + 192B^2M^2 \underset{0}{p_{34}} \underset{0}{\rho}$$

$$- 8BM \underset{0}{p_{12}} \underset{0}{\rho^2} + 24BM \underset{0}{p_{34}^2} - 192BM \underset{0}{p_{12}} \underset{0}{p_{34}} \underset{0}{\rho}$$

$$+ 32BM \underset{0}{p_{34}^3} - 240BM \underset{0}{p_{12}} \underset{0}{p_{34}^2} - 12 \underset{0}{p_{12}} \underset{0}{p_{34}^2} \underset{0}{\rho}$$

$$+ 48 \underset{0}{p_{12}^2} \underset{0}{p_{34}} \underset{0}{\rho} - 16 \underset{0}{p_{12}} \underset{0}{p_{34}^3}. \tag{A29}$$

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Some generating functions of modified Gegenbauer polynomials

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Abstract. L Weisner's group theoretic method has been introduced in the study of special function. In this paper we obtain two differential operators, one of which simultaneously raises the index and lowers the parameter of modified Gegenbauer polynomials $C_n^{\nu+n}(x)$ by unity and the other acts on $C_n^{\nu+n}(x)$ in the reversed way by suitable interpretation to the index n and the parameter ν of $C_n^{\nu+n}(x)$. We have also found out the extended form of the groups generated by the operators $A_{ij}(i,j=1,2)$. We have also derived some novel generating functions of $C_n^{\nu+n}(x)$ from which several special generating functions can be easily derived.

Keywords. Generating functions; Gegenbauer polynomials.

1. Introduction

The Gegenbauer polynomials $C_n^{\nu}(x)$ defined by [3]

$$C_n^{\nu}(x)=\sum_{k=0}^{(n/2)}(-1)^k\frac{(\nu)_{n-k}(2x)^{n-2k}}{k!(n-2k)!} \tag{1}$$

is a solution of the following ordinary differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-(2\nu+1)x\frac{dy}{dx}+n(2\nu+n)y=0. \tag{2}$$

In this note we consider the modified Gegenbauer polynomials $[e_n^{\nu+n}(x)]$ which satisfy the following ordinary differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-(2\nu+2n+1)x\frac{dy}{dx}+n(2\nu+3n)y=0. \tag{3}$$

Several generating functions for Gegenbauer polynomials have been derived by different methods namely classical, theory of Lie groups etc. McBride [2] has derived some generating functions for Gegenbauer polynomials by interpreting the index n of the said polynomials with the help of Weisner's [4] group theoretic method. Chongdar [1] studied the Gegenbauer polynomials $C_n^{\nu}(x)$ to obtain generating functions simultaneously by means of suitable interpretations to the index n and the parameter ν of $C_n^{\nu}(x)$ instead of n only. The object of the present paper is to derive some generating functions of modified Gegenbauer polynomials $C_n^{\nu+n}(x)$ instead of modified Gegenbauer polynomials $C_n^{\nu-n}(x)$ by constructing a Lie algebra with the aid of Weisner's

[2] as a result of single interpretation to the index n of the polynomial $C_n^{\nu}(x)$. The main results of our investigation are given in §3.

2. Group-theoretic discussion

Replacing d/dx by $\partial/\partial x$, n by $y(\partial/\partial y)$, ν by $z(\partial/\partial z)$ and y by $u(x, y, z)$ in (3) we get the following partial differential equation

$$(1-x^2)\frac{\partial^2 u}{\partial x^2} - 2xz\frac{\partial^2 u}{\partial z \partial x} - 2xy\frac{\partial^2 u}{\partial y \partial x} + 2yz\frac{\partial^2 u}{\partial y \partial z}$$

$$+ 3y^2\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial x} + 3y\frac{\partial u}{\partial y} = 0. \tag{4}$$

Thus we see that

$$U_1(x, y, z) = C_n^{\nu+n}(x)y^n z^{\nu}$$

is a solution of (4) since $C_n^{\nu+n}(x)$ is a solution of (3).

Now by using the following differential recurrence relations

$$(x^2-1)DC_n^{\nu+n}(x) = -(2\nu+3n-1)C_{n-1}^{\nu+n}(x) - nxC_n^{\nu+n}(x) \tag{5}$$

and

$$(x^2-1)DC_n^{\nu+n}(x) = (n+1)C_{n+1}^{\nu+n}(x) - (2\nu+3n)xC_n^{\nu+n}(x) \tag{6}$$

we define the infinitesimal operators $A_{ij}(i=1,2; j=1,2)$

$$A_{ij} = A_{ij}^{(1)}\frac{\partial}{\partial x} + A_{ij}^{(2)}\frac{\partial}{\partial y} + A_{ij}^{(3)}\frac{\partial}{\partial z} + A_{ij}^{(0)} \quad (i,j=1,2)$$

as follows

$$A_{11} = y\frac{\partial}{\partial y}; \quad A_{12} = (x^2-1)y^{-1}z\frac{\partial}{\partial x} - xz\frac{\partial}{\partial y}; \quad A_{21} = z\frac{\partial}{\partial z}$$

and

$$A_{22} = (x^2-1)yz^{-1}\frac{\partial}{\partial x} + 3xy^2z^{-1}\frac{\partial}{\partial y} + 2xy\frac{\partial}{\partial z}$$

such that

$$A_{11}(C_n^{\nu+n}(x)y^n z^{\nu}) = nC_n^{\nu+n}(x)y^n z^{\nu},$$

$$A_{12}(C_n^{\nu+n}(x)y^n z^{\nu}) = -(2\nu+3n-1)C_{n-1}^{(\nu+1)+(n-1)}(x)y^{n-1}z^{\nu+1},$$

$$A_{21}(C_n^{\nu+n}(x)y^n z^{\nu}) = \nu C_n^{\nu+n}(x)y^n z^{\nu},$$

$$A_{22}(C_n^{\nu+n}(x)y^n z^{\nu}) = (n+1)C_{n+1}^{(\nu-1)+(n+1)}(x)y^{n+1}z^{\nu-1}.$$

$$[A, B]u = (AB - BA)u,$$

we have

$$[A_{11}, A_{12}] = A_{12}, \quad [A_{11}, A_{22}] = A_{22}, \quad [A_{12}, A_{22}] = -2(A_{11} + A_{21}),$$

$$[A_{11}, A_{21}] = 0, \quad [A_{12}, A_{21}] = -A_{12}, \quad [A_{21}, A_{22}] = -A_{22}. \tag{7}$$

From the above commutator relations we can easily state the following theorem.

THEOREM. The set of linear combinations of operators A_{11}, A_{12}, A_{21} and A_{22} forms a Lie algebra.

It can be easily shown that the partial differential operator L given by

$$L = (1 - x^2)\frac{\partial^2}{\partial x^2} - 2xz\frac{\partial^2}{\partial z \partial x} - 2xy\frac{\partial^2}{\partial y \partial x} + 2yz\frac{\partial^2}{\partial y \partial z}$$

$$+ 3y^2\frac{\partial^2}{\partial y^2} - x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$$

can be expressed as follows

$$(1 - x^2)L = (A_{11} + 1)(2A_{21} + 3A_{11}) + A_{12}A_{22}. \tag{8}$$

It can be easily verified that $(1 - x^2)L$ commutes with $A_{ij}(i, j = 1, 2)$ i.e.

$$[(1 - x^2)L, A_{ij}] = 0 \quad (i, j = 1, 2). \tag{9}$$

The extended form of the groups generated by $A_{ij}(i, j = 1, 2)$ are given by

$$\exp(a_{11}A_{11})f(x, y, z) = f(x, \exp(a_{11})y, z), \tag{10}$$

$$\exp(a_{12}A_{12})f(x, y, z) = f\left(\frac{xy - a_{12}z}{(a_{12}^2 z^2 - 2a_{12}xyz + y^2)^{1/2}},\right.$$

$$\left.(a_{12}^2 z^2 - 2a_{12}xyz + y^2)^{1/2}, z\right) \tag{11}$$

$$\exp(a_{21}A_{21})f(x, y, z) = f[x, y, \exp(a_{21})z], \tag{12}$$

$$\exp(a_{22}A_{22})f(x, y, z) = f\left(\frac{xz - a_{22}y}{(a_{22}^2 y^2 - 2a_{22}xyz + z^2)^{1/2}},\right.$$

$$\frac{yz^3}{(a_{22}^2 y^2 - 2a_{22}xyz + z^2)^{3/2}},$$

$$\left.\frac{z^3}{a_{22}^2 y^2 - 2a_{22}xyz + z^2}\right).$$

From the above relations we get

$$\exp(a_{22}A_{22})\exp(a_{12}A_{12})\exp(a_{21}A_{21})\exp(a_{11}A_{11})f(x,y,z)=f(\xi,\eta,\zeta), \tag{14}$$

where

$$\xi=\frac{y(xz-a_{22}y)-a_{12}(a_{22}^2y^2-2a_{22}xyz+z^2)}{(a_{22}^2y^2-2a_{22}xyz+z^2)^{1/2}[a_{12}^2(a_{22}^2y^2-2a_{22}xyz+z^2)-2a_{12}y(xz-a_{22}y)+y^2]^{1/2}}$$

$$\eta=\exp(a_{11})\frac{z^3[a_{12}^2(a_{22}^2y^2-2a_{22}xyz+z^2)-2a_{12}y(xz-a_{22}y)+y^2]^{1/2}}{(a_{22}^2y^2-2a_{22}xyz+z^2)^{3/2}}$$

$$\zeta=\exp(a_{21})\frac{z^3}{a_{22}^2y^2-2a_{22}xyz+z^2}.$$

3. Generating functions

From (4), $u(x,y,z)=C_n^{\nu+n}(x)y^nz^\nu$ is a solution of the system

$$\begin{array}{lll} Lu=0 & Lu=0 & Lu=0 \\ (A_{11}-n)u=0\,; & (A_{21}-\nu)u=0\,; & (A_{11}+A_{21}-n-\nu)u=0\,. \end{array}$$

From (9) we easily get

$$S(1-x^2)L(C_n^{\nu+n}(x)y^nz^\nu)=(1-x^2)LS(C_n^{\nu+n}(x)y^nz^\nu)=0,$$

where

$$S=\exp(a_{22}A_{22})\exp(a_{12}A_{12})\exp(a_{21}A_{21})\exp(a_{11}A_{11}).$$

Therefore the transform $S(C_n^{\nu+n}(x)y^nz^\nu)$ is annulled by $(1-x^2)L$. Now putting $a_{11}=a_{21}=0$ and replacing $f(x,y,z)$ by $C_n^{\nu+n}(x)y^nz^\nu$ in (14) we get

$$\begin{aligned}&[\exp(a_{22}A_{22})\exp(a_{12}A_{12})](C_n^{\nu+n}(x)y^nz^\nu)\\&=z^{3n+3\nu}[a_{12}^2(a_{22}^2y^2-2a_{22}xyz+z^2)-2a_{12}y(xz-a_{22}y)+y^2]^{n/2}\\&\times(a_{22}^2y^2-2a_{22}xyz+z^2)^{-((3n/2)+\nu)}\times C_n^{\nu+n}\\&\times\left(\frac{y(xz-a_{22}y)-a_{12}(a_{22}^2y^2-2a_{22}xyz+z^2)}{(a_{22}^2y^2-2a_{22}xyz+z^2)^{1/2}[a_{12}^2(a_{22}^2y^2-2a_{22}xyz+z^2)-2a_{12}y(xz-a_{22}y)+y^2]^{1/2}}\right).\end{aligned} \tag{15}$$

On the other hand we get

$$\begin{aligned}&[\exp(a_{22}A_{22})\exp(a_{12}A_{12})](C_n^{\nu+n}(x)y^nz^\nu)\\&=\sum_{m=0}^{\infty}\frac{(a_{22})^m}{m!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}(n-k+1)_m(-2\nu-3n+1)_kC_{n-k+m}^{\nu+n}\end{aligned}$$

$$\times (a_{22}^2 y^2 - 2a_{22}xyz + z^2)^{-(3n/2)-\nu} \times C_n^{\nu+n}$$

$$\times \left(\frac{y(xz - a_{22}y) - a_{12}(a_{22}^2 y^2 - 2a_{22}xyz + z^2)}{(a_{22}^2 y^2 - 2a_{22}xyz + z^2)^{1/2}[a_{12}^2(a_{22}^2 y^2 - 2a_{22}xyz + z^2) - 2a_{12}y(xz - a_{22}y) + y^2]^{1/2}} \right)$$

$$= \sum_{m=0}^{\infty} \frac{(a_{22})^m}{m!} \sum_{k=0}^{n} \frac{(a_{12})^k}{k!} (n-k+1)_m (-2\nu - 3n + 1)_k C_{n-k+m}^{\nu+n}(x) y^{n-k+m} z^{\nu+k-m}. \tag{17}$$

The above generating relation which yields a good number of particular generating relations by attributing different values to $a_{ij} (i = 1, 2; j = 2)$ seems to be new.

4. Derivation of some interesting results as the particular case of (17)

Case 1. Putting $a_{12} = 1$, $a_{22} = 0$ and writing $yz^{-1} = t$ in (17) we get

$$(t^2 - 2xt + 1)^{n/2} C_n^{\nu+n}\left(\frac{xt - 1}{(t^2 - 2xt + 1)^{1/2}} \right)$$

$$= \sum_{k=0}^{n} \frac{(1 - 2\nu - 3n)_k}{k!} C_{n-k}^{\nu+n}(x) t^{n-k}. \tag{18}$$

Case 2. Letting $a_{22} = 1, a_{12} = 0$ and writing $yz^{-1} = t$ in (17) we easily obtain

$$(1 - 2xt + t^2)^{-(3n/2)-\nu} C_n^{\nu+n}\left(\frac{x - t}{(1 - 2xt + t^2)^{1/2}} \right)$$

$$= \sum_{m=0}^{\infty} \binom{n+m}{m} C_{n+m}^{\nu+n}(x) t^m. \tag{19}$$

Case 3. Putting $a_{12} = (-1/w)$, $a_{22} = 1$ and substituting $yz^{-1} = t$, we get from (17)

$$\left\{ \frac{1}{w^2}(1 - 2xt + t^2) + \frac{2t}{w}(x - t) + t^2 \right\}^{n/2} (1 - 2xt + t^2)^{-(3n/2)-\nu}$$

$$\times C_n^{\nu+n}\left(\frac{t(x - t) + \frac{1}{w}(1 - 2xt + t^2)}{(1 - 2xt + t^2)^{1/2}\left[\frac{1}{w^2}(1 - 2xt + t^2) + \frac{2t}{w}(x - t) + t^2 \right]^{1/2}} \right)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{w} \right)^k}{k!} (n-k+1)_m (1 - 2\nu - 3n)_k C_{n-k+m}^{\nu+n}(x) t^{n-k+m}. \tag{20}$$

Case 4. Since $[A_{12}, A_{22}] \neq 0$, now applying the operator $\exp(a_{12}A_{12})\exp(a_{22}A_{22})$ on the function $C_n^{\nu+n}(x)y^n z^\nu$ we obtain the following generating relations which is different from (17)

$$(w^2 - 2a_{12}xw + a_{12}^2)^{n/2}\{a_{22}^2 w^2 - 2a_{22}xw(1 + a_{12}a_{22}) + (1 + a_{12}a_{22})^2\}^{-(3n/2)-\nu}$$

$$\times C_n^{\nu+n}\left(\frac{(xw - a_{12}) - a_{22}(w^2 - 2a_{12}xw + a_{12}^2)}{(w^2 - 2a_{12}xw + a_{12}^2)^{1/2}[a_{22}^2 w^2 - 2a_{22}xw(1 + a_{12}a_{22}) + (1 + a_{12}a_{22})^2]^{1/2}}\right).$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{n+m}\frac{(a_{22})^m}{m!}\frac{(a_{12})^k}{k!}(n+1)_m(-2\nu - 3m - 5n + 1)_k C_{n-k+m}^{\nu+n}(x)w^{n-k+m}, \tag{21}$$

on substituting $yz^{-1} = w$.

Subcase of case 1. Now replacing ν by $\nu - n$ and writing $t^{-1} = t_1$ in (18) we get

$$(t_1^2 - 2xt_1 + 1)^{n/2} C_n^\nu\left(\frac{x - t_1}{(t_1^2 - 2xt_1 + 1)^{1/2}}\right) = \sum_{k=0}^{n}\frac{(1 - 2\nu - n)_k}{k!}C_{n-k}^\nu(x)t_1^k, \tag{22}$$

which is due to McBride [2, p. 52].

Subcase of case 2. Again on replacing ν by $\nu - n$ we obtain from (19)

$$(1 - 2xt + t^2)^{-\nu-(n/2)} C_n^\nu\left(\frac{x - t}{(1 - 2xt + t^2)^{1/2}}\right) = \sum_{m=0}^{\infty}\binom{n+m}{m}C_{n+m}^\nu(x)t^m, \tag{23}$$

which may be compared with the result of McBride [2, p. 52].

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Unified theorems involving H-function transform and Meijer Bessel function transform

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Abstract. A theorem which reveals an interesting relationship between originals of related functions in H-function transform and Meijer Bessel function transforms is established. Another theorem interconnecting the two Meijer Bessel function transforms is also obtained. These theorems seem to be capable of unifying and extending a large number of results obtained by earlier workers. Finally, we evaluate an integral by way of illustration.

Keywords. H-function transform; Meijer Bessel function transform.

1. Introduction

Theorem 1 established in this paper exhibits an interesting relationship between the originals of the related functions in H-function transform and the Meijer Bessel function transform. Theorem 2 exhibits the relationship between originals of related functions in the Meijer Bessel function transforms. The earlier theorems [3, 2, 8] follow as particular cases of our findings. Several useful integrals involving various special functions can be evaluated as an application of our findings. We evaluate here only one integral by way of illustration giving us the Laplace transform of the function involving $S_2(\frac{1}{4} \pm (\omega/2), \frac{1}{4} \pm (\mu/2); at^{1/2})$ which can find application in solving boundary value problems.

2. Results

The following results obtained from the known results [5] will be required in the sequel

(I)
$$K\left\{2^{(1/2)-c}x^c H^{m,n}_{p,q+2}\left[t\left(\frac{x}{2}\right)^{\sigma x}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right];\omega;s\right\}$$
$$= s^{-c-1}H^{m,n}_{p,q}\left[ts^{-\sigma}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] \tag{1}$$

provided

$$\sigma > 0,\quad \mathrm{Re}(s) > 0,\quad \min\left[\mathrm{Re}\left\{\frac{b_j}{\beta_j}(j=1,\ldots,m)\right\}, \mathrm{Re}\left\{\left(c+\omega+\frac{3}{2}\right)\right\}\right] > 0$$

$$\sum_{j=1} \beta_j - \sum_{j=m+1} \beta_j + \sum_{j=1} \alpha_j - \sum_{j=n+1} \alpha_j - \sigma_x > 0.$$

(II)
$$K\left\{2^{-c}x^c H_{0,4}^{2,0}\left[\frac{t^2}{4}\left(\frac{x}{2}\right)^{2\sigma}\middle|\begin{matrix}\\ \left(\frac{1}{4}\pm\frac{\mu}{2},1\right),\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\sigma\right)\end{matrix}\right];\omega;s\right\}$$

$$= s^{-c-1}(ts^{-\sigma})^{1/2}K_{\mu}(ts^{-\sigma}) \tag{2}$$

provided

$$0<\sigma\leqslant 1, \quad \mathrm{Re}(s)>0 \quad \text{and} \quad \min\left[\mathrm{Re}\left\{\sigma\left(\tfrac{1}{2}\pm\mu\right)+\left(c\pm\omega+\tfrac{3}{2}\right)\right\}\right]>0,$$

where the H-function involved in each of the above equations stands for the well-known Fox's H-function which is quite general in nature and includes most of the commonly used functions as its special cases. The definition of H-function in terms of Mellin–Barnes type of contour integral, the various conditions satisfied by its parameters for the convergence of the defining integral, some of its special cases, the important properties and the asymptotic expansion can be referred to in a standard work on this function [7]. It has been assumed that the various H-functions occurring in this paper satisfy the conditions corresponding appropriately to those given by (2.2.11) on page 13, in the book referred above.

3. Main theorems

THEOREM 1.
If

$$h(s) = H\{f(x);s\} = \int_0^{\infty} H_{p,q}^{m,n}\left[sx\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]f(x)\,\mathrm{d}x \tag{3}$$

and

$$s^{-c-1}h(s^{-\sigma}) = K\{g(x);\omega;s\} = \int_0^{\infty}(sx)^{1/2}K_{\omega}(sx)g(x)\,\mathrm{d}x, \tag{4}$$

then

$$g(x) = 2^{(1/2)-c}x^c\int_0^{\infty} H_{p,q+2}^{m,n}$$

$$\times\left[t\left(\frac{x}{2}\right)^{\sigma}\middle|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right]f(t)\,\mathrm{d}t \tag{5}$$

provided that the integrals involved in (3) to (5) are absolutely convergent,

$$\sigma>0, \quad \mathrm{Re}(s)>0, \quad \min\left[\mathrm{Re}\left\{\frac{b_j}{\beta_j}(j=1,\ldots,m)+\left(c\pm\omega+\tfrac{3}{2}\right)\right\}\right]>0$$

$$\sum_{j=1}^{p} \beta_j + \sum_{j=m+1}^{q} \beta_j + \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j > 0,$$

where the transform occurring in (3) was introduced by Gupta and Mittal [7] and is known as the H-function transform while the transform occurring in (4) was introduced by Meijer [6] and is known as the Meijer Bessel function transform.

Proof. From (3), we easily obtain

$$s^{-c-1}h(s^{-\sigma}) = \int_0^\infty s^{-c-1} H_{p,q}^{m,n}\left[ts^{-\sigma} \middle| \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] f(t)\,dt. \tag{6}$$

Now with the help of (1) we get

$$\begin{aligned} s^{-c-1}h(s^{-\sigma}) &= \int_0^\infty K\left\{ 2^{(1/2)-c} x^c H_{p,q+2}^{m,n} \right. \\ &\quad \left. \times\left[t\left(\frac{x}{2}\right)^{\sigma} \middle| \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right]; \omega; s \right\} f(t)\,dt \\ &= \int_0^\infty \left\{ \int_0^\infty (sx)^{1/2} K_\omega(sx)\, 2^{(1/2)-c} x^c H_{p,q+2}^{m,n} \right. \\ &\quad \left. \times\left[t\left(\frac{x}{2}\right)^{\sigma} \middle| \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right] dx \right\} f(t)\,dt. \end{aligned} \tag{7}$$

On inverting the order of integration in (7) which is easily seen to be permissible under the conditions stated with the theorem, we get

$$\begin{aligned} s^{-c-1}h(s^{-\sigma}) &= \int_0^\infty (sx)^{1/2} K_\omega(sx) \left\{ \int_0^\infty 2^{(1/2)-c} x^c H_{p,q+2}^{m,n} \right. \\ &\quad \left. \times\left[t\left(\frac{x}{2}\right)^{\sigma} \middle| \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},\left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2},\frac{\sigma}{2}\right)\end{matrix}\right] f(t)\,dt \right\} dx. \end{aligned} \tag{8}$$

Now comparing (8) with (4), we get the required result (5).

Now, if we take both the transforms to be Meijer Bessel function transforms defined earlier by (4) and proceeding along the lines followed in Theorem 1 and using the result (2) we arrive at the following theorem.

THEOREM 2.

If

$$h(s) = K\{f(x);\mu;s\} = \int_0^\infty (sx)^{1/2} K_\mu(sx) f(x)\,dx, \tag{9}$$

$$s^{-1}h(s^{-1}) = K\{g(x); \omega; s\} = \int_0^\infty (sx)^{-\omega} K_\omega(sx) g(x)\,dx, \tag{10}$$

then

$$g(x) = 2^{-c}x^c \int_0^\infty H_{0,4}^{2,0}\left[\frac{t^2}{4}\left(\frac{x}{2}\right)^{2\sigma}\middle|\begin{matrix}\\ \end{matrix}\left(\frac{1}{4}\pm\frac{\mu}{2}, 1\right), \left(\frac{1}{4}-\frac{c}{2}\pm\frac{\omega}{2}, \sigma\right)\right] f(t)\,dt \tag{11}$$

provided that the integrals involved in (9) to (11) are absolutely convergent, $0 < \sigma \leqslant 1$, $\mathrm{Re}\,(s) > 0$ and $\min\left[\mathrm{Re}\left\{\sigma(\frac{1}{2}\pm\mu) + (c \pm \omega + \frac{3}{2})\right\}\right] > 0$.

Special cases. If in Theorem 2, $\sigma = (n/s)$, then by simplifying and using a known result of [7], we get a known theorem [3]. Also, substituting $\mu = \omega = \frac{1}{2}$ in Theorem 2 and simplifying we get a theorem of [2]. Further, if $\sigma = 1$ and $\mu = \omega = \frac{1}{2}$ in Theorem 2, we get a well-known Tricomi's theorem [8].

4. Application

In Theorem 2, take $\sigma = 1$, $c = 0$ and let

$$h(s) = \tfrac{1}{2}\alpha^k s^{-1/2}\Gamma\left(\frac{1+\mu}{2}-k\right)\Gamma\left(\frac{1-\mu}{2}-k\right)\exp\left(\frac{s^2}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^2}{4\alpha}\right). \tag{12}$$

From the known result [1], we have

$$\tfrac{1}{2}\alpha^k s^{-1/2}\Gamma\left(\frac{1+\mu}{2}-k\right)\Gamma\left(\frac{1-\mu}{2}-k\right)\exp\left(\frac{s^2}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^2}{4\alpha}\right)$$

$$= K\{x^{-(1/2)-2k}\exp(-\alpha x^2); \mu; s\}, \tag{13}$$

where $\mathrm{Re}\,(\alpha) > 0$, $2\,\mathrm{Re}\,(k) < 1 - |\mathrm{Re}\,(\mu)|$.

Now interpreting (12) and (13) in conjunction with (9), we get

$$f(x) = x^{-(1/2)-2k}\exp(-\alpha x^2). \tag{14}$$

Again, from (12) we have

$$s^{-1}h(s^{-1}) = \tfrac{1}{2}\alpha^k s^{-1/2}\Gamma\left(\frac{1+\mu}{2}-k\right)\Gamma\left(\frac{1-\mu}{2}-k\right)$$

$$\times \exp\left(\frac{s^{-2}}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^{-2}}{4\alpha}\right). \tag{15}$$

Also from another known result [4], by taking $p = s$, $\sigma = \frac{1}{2}$, $n = s = 1$, $a = 1/4\alpha$, $m = \frac{1}{2}\mu$ therein, we get after some simplification

$$\tfrac{1}{2}\alpha^k s^{-1/2}\Gamma(\tfrac{1}{2}-k+\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}-k-\tfrac{1}{2}\mu)\exp\left(\frac{s^{-2}}{8\alpha}\right)W_{k,\frac{\mu}{2}}\left(\frac{s^{-2}}{4\alpha}\right)$$

$$= K\left\{\alpha^k x^{-1/2} G^{2,1}_{1,4}\left[\frac{x^2}{16\alpha}\middle|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right]; s\right\}, \tag{16}$$

where $\mathrm{Re}(2 \pm \mu \pm \omega) > 0$, $\mathrm{Re}(s) > 0$. Now interpreting (15) and (16) in conjunction with (10), we get

$$g(x) = \alpha^k x^{-1/2} G^{2,1}_{1,4}\left[\frac{x^2}{16\alpha}\middle|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right]. \tag{17}$$

Finally, from (11), (14) and (17), we get after some simplification the following integral

$$\int_0^\infty t^{-(3/2)-2k}\exp(-\alpha t^2) S_2\left(\frac{1}{4}\pm\frac{\mu}{2},\frac{1}{4}\pm\frac{\omega}{2};\frac{tx}{4}\right)\mathrm{d}t$$

$$= \frac{\alpha^k}{4} x^{1/2} G^{2,1}_{1,4}\left[\frac{x^2}{16\alpha}\middle|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right], \tag{18}$$

where $2\,\mathrm{Re}(k) < 1 - |\mathrm{Re}(\mu)|$, $\mathrm{Re}(\alpha) > 0$.

If in the above integral, we take $\alpha = p, (x/4) = a$ and change the variable of integration slightly, we obtain the following result

$$L\left\{t^{-k-(5/4)} S_2\left(\frac{1}{4}\pm\frac{\mu}{2},\frac{1}{4}\pm\frac{\omega}{2};at^{1/2}\right);p\right\}$$

$$= \int_0^\infty t^{-k-(5/4)} S_2\left(\frac{1}{4}\pm\frac{\mu}{2},\frac{1}{4}\pm\frac{\omega}{2};at^{1/2}\right)\exp(-pt)\,\mathrm{d}t$$

$$= a^{1/2} p^k G^{2,1}_{1,4}\left[\frac{a^2}{p}\middle|\begin{matrix}1+k\\ \frac{1\pm\mu}{2},\frac{1\pm\omega}{2}\end{matrix}\right], \tag{19}$$

where $2\,\mathrm{Re}(k) < 1 - |\mathrm{Re}(\mu)|$, $\mathrm{Re}(p) > 0$. The transform occurring in (19) is the well-known Laplace transform.

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On focal locus of submanifolds of naturally reductive compact Riemannian homogeneous spaces

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Abstract. The purpose of this paper is to discuss focal points of submanifolds of naturally reductive compact Riemannian homogeneous spaces. Focal points have been studied by other workers [1, 4, 7]. In this paper we have estimated location of focal points along geodesic and obtained a condition under which a point is a focal point along geodesic.

Keywords. Focal locus; submanifolds; Riemannian homogeneous spaces.

1. Introduction

Let M be a C^∞, n-dimensional compact Riemannian manifold with metric $\langle , \rangle$. Let L be a compact connected submanifold embedded in M. The orthogonal complement N_pL of the tangent space T_pL in T_pM will be called the normal space of L at p for $p \in L$. The union $N = \cup \{N_pL | p \in L\}$ of these normal spaces which is a subbundle of class C^∞ of the tangent bundle TM is called the normal bundle of the submanifold L.

Let $\gamma: [0, \beta] \to M$, $\beta \in R$ is a normal geodesic with $\gamma(0) = z \in L, \dot{\gamma}(0) = v \in N_pL$.

Let $E_1, \ldots, E_{n-1}: [0, \beta] \to TM$ be D-parallel vector fields along γ such that $(\dot{\gamma}(\tau), E_1(\tau), \ldots, E_{n-1}(\tau))$ is an orthonormal base for $T_{\gamma(\tau)}M$.

Let $Y(\tau), \tau \in [0, \beta]$ be a Jacobi field along γ. If T_zL is spanned by $E_1(0), \ldots, E_k(0)$, $Y(\tau)$ is an L-Jacobi-field if [1].

(i) $\langle Y(\tau), \dot{\gamma}(\tau) \rangle \equiv 0$,
(ii) $Y(0) \in T_zL$,
(iii) $D_{\dot{\gamma}(0)} Y(0) + \sigma_{\dot{\gamma}(0)} Y(0) \in N_zL$,

where $\sigma_{\dot{\gamma}(0)}: T_zL \to T_zL$ is a Weingarten map, and D is the connection in the case when M is a naturally reductive homogeneous space which is considered in § 2.

A point $\gamma(\tau_0)$ on γ, for $\tau_0 \in (0, \beta]$, is called a focal point of L with respect to γ if there is a L-Jacobi field on γ that is not identically zero, that vanishes at $\tau = \tau_0$. The set of such focal points is called focal locus of L in M.

In this paper we will give estimates of the location of focal points by using the Jacobi field of the form given in [8].

From a classical point of view it would actually be more appropriate to carry out the computations as in this paper with respect to the Levi–Civita connection ∇ rather than D as we have done. But we are unable to do this at present. We note however that if the

The Riemannian manifold M is called homogeneous if the isometry group of $\langle , \rangle$ acts transitively on M. If G is a group acting transitively by isometries on M we can write $M = G/H$ where H is the isotropy subgroup of G at fixed point p_0. We denote the Lie algebra of G, H by $\mathfrak{g}, \mathfrak{h}$. M is called reductive homogeneous if there exists a complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ so that $\mathrm{Ad}(H)$ leaves $\mathfrak{m}$ invariant. Since H is compact this is always the case but notice that $\mathfrak{m}$ is not necessarily unique. $\mathfrak{m}$ can be identified with $T_{p_0}M$. The metric on $T_{p_0}M$ thus induces a metric on $\mathfrak{m}$ again denoted by $\langle , \rangle$. M being reductive implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ [3], [5].

If $X, Y \in \mathfrak{m}$ then we denote by $[X, Y]_{\mathfrak{h}}$, $[X, Y]_{\mathfrak{m}}$ the $\mathfrak{h}$ and $\mathfrak{m}$ component of $[X, Y]$. If $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ then M is symmetric.

M is called naturally reductive if

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0,$$

for all $X, Y, Z \in \mathfrak{m}$. In other words if $[X, \cdot]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric for all $X \in \mathfrak{m}$.

We will denote the Levi–Civita connection and the curvature of $\langle , \rangle$ by ∇ and R. If M is naturally reductive the connection ∇ can be described as follows:

If X is a Killing vector field and $v \in T_{p_0}M \simeq \mathfrak{m}$ then

$$\nabla_v X = \begin{cases} [X, v] & \text{if } X \in \mathfrak{h} \\ \frac{1}{2}[X, v]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m}. \end{cases}$$

The curvature tensor at p_0 is given by $R(X, v)v = -[v, [v, X]_{\mathfrak{h}}] - \frac{1}{4}[v, [v, X]_{\mathfrak{m}}]_{\mathfrak{m}}$.

One knows that on a naturally reductive space there exists a connection D with torsion T and curvature B so that T and B are D parallel. D has the same geodesics as ∇ so that

$$\nabla_X Y = D_X Y - \tfrac{1}{2} T(X, Y),$$

and D, T and B at p_0 can be expressed in terms of the Lie bracket:

$$D_v X = \begin{cases} [X, v] & \text{if } X \in \mathfrak{h} \\ [X, v]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m}, \end{cases}$$

$$T(X, Y) = -[X, Y]_{\mathfrak{m}},$$

$$B(X, Y)Z = -[[X, Y]_{\mathfrak{h}}, Z],$$

where $X, Y, Z \in \mathfrak{m}$. Notice that $R = B - \frac{1}{4}T^2$.

Each $\alpha \in G$ operates on M by left multiplication $L_\alpha : G/H \to G/H$. The maps L_α are isometries.

The geodesics in a naturally reductive space are images of one parameter groups in G.

Let $\exp_G$ be the exponential map of G. Then for $v \in \mathfrak{m}$, $L_{\exp_G \tau v} \cdot p_0$ is the geodesic through p_0. The derivative of $L_{\exp_G \tau v}$ at p_0 is parallel translation along $L_{\exp_G \tau v} \cdot p_0$ with respect to the connection D. Since ∇ and D have the same geodesics they also have the same Jacobi fields. But the Jacobi equation with respect to D along $\gamma(\tau) = L_{\exp_G \tau v} \cdot p_0$,

is much simpler since T and B are D parallel. If we write X as $X(\tau) = d(L_{\exp_G \tau v}) p_0(Y(\tau))$, then the Jacobi equation reads:

$$Y'' - T(Y') + B(Y) = 0,$$

where

$$T(Y) = T(v, Y) = -[v, Y]_m,$$

$$B(Y) = B(Y, v)v = -[v, [v, Y]_f].$$

This is a differential equation in the vector space with constant coefficients, T is skew-symmetric and B is symmetric. The solutions of this equation are obtained by substituting $Y(\tau) = A(\tau) \cdot \exp(m\tau)$, where m is a complex number and $A(\tau)$ a complex vector valued polynomial. The real and complex parts of these solutions then give a basis of the Jacobi fields along γ.

We shall use the following theorem given by Ziller [8].

THEOREM: If one solves the Jacobi equation on a compact naturally reductive Riemannian homogeneous space in the form $Y(\tau) = A(\tau) \cdot \exp(m\tau)$ with $A(\tau)$ a polynomial with D parallel complex vector fields as coefficients and a complex number, one has:

(i) m is imaginary or 0;

(ii) if m is imaginary and $\neq 0$, then $A(\tau)$ is a constant polynomial so that the corresponding Jacobi fields are of the form:

$$Y(\tau) = \operatorname{Re} A \cos a\tau - \operatorname{Im} A \sin a\tau,$$

$$Y(\tau) = \operatorname{Re} A \sin a\tau + \operatorname{Im} A \cos a\tau,$$

with $m = i \cdot a$ and $A(\tau) = A\, a D$ parallel vector field with $(m^2 Id - mT + B)A = 0$;

(iii) if $m = 0$, then $A(\tau) = A_1 \tau + A_0$ with A_1 and $A_0 D$ parallel (real) vector fields are the only possible Jacobi fields where $B(A_1) = 0$ and $B(A_0) = T(A_1)$.

3. Focal locus of submanifold

Our first aim is to specify the form of Jacobi field given in the above theorem so as to give an L-Jacobi field. Since parallel vector fields $E_1, \ldots, E_{n-1}$ with $\dot{\gamma}(\tau)$ form an orthonormal base for $T_{\gamma(\tau)}M$, $Y(\tau)$ defined along γ can be expressed as

$$Y(\tau) = \sum_{i=1}^{n-1} (\alpha_i(\tau) \cos a\tau - \beta_i(\tau) \sin a\tau) E_i(\tau),$$

$$Y(\tau) = \sum_{i=1}^{n-1} (\alpha_i(\tau) \sin a\tau + \beta_i(\tau) \cos a\tau) E_i(\tau),$$

where $\alpha_i(\tau)$ and $\beta_i(\tau)$ are real and imaginary parts of $A(\tau)$ respectively, where $A(\tau)$ is a constant polynomial corresponding to (ii) of the above theorem.

$$Y(0)=\sum_{i=1}^{k}\beta_i(0)\,E_i(0)\in T_zL$$

and therefore $\beta_{k+1}=\cdots=\beta_{n-1}=0$, if $(E_1,\ldots,E_k)$ is base for T_zL.

For condition (iii), we have

$$D_{\dot\gamma(\tau)}Y(\tau)=\sum_{i=1}^{n-1}(-\alpha_i a\sin a\tau-\beta_i a\cos a\tau)E_i(\tau)$$

since E_i are D-parallel. Therefore

$$D_{\dot\gamma(0)}Y(0)=\sum_{i=1}^{n-1}-a\,\beta_i E_i(0)$$

and

$$\sigma_{\dot\gamma(0)}Y(0)=\sigma_{\dot\gamma(0)}\sum_{i=1}^{k}\alpha_i E_i(0)=\sum_{i=1}^{k}\alpha_i\lambda_i\,E_i(0),$$

where λ_i are nonzero eigenvalues of symmetric linear transformation $\sigma_{\dot\gamma(0)}$ appli
eigenvectors E_i. Therefore

$$D_{\dot\gamma(0)}Y(0)+\sigma_{\dot\gamma(0)}Y(0)=\sum_{i=1}^{n-1}-a\beta_i E_i+\sum_{i=1}^{k}\alpha_i\lambda_i E_i\in N_zL$$

if

$$-a\beta_i+\alpha_i\lambda_i=0\quad\text{for } i=1,\ldots,k.$$

Similarly for the second form of $Y(\tau)$ we have

$$D_{\dot\gamma(0)}Y(0)+\sigma_{\dot\gamma(0)}Y(0)=\sum_{i=1}^{n-1}a\alpha_i E_i+\sum_{i=1}^{k}\beta_i\lambda_i E_i\in N_zL$$

if

$$a\,\alpha_i+\beta_i\lambda_i=0\quad\text{for } i=1,\ldots,k.$$

So, the general form of L-Jacobi fields is of the form

$$Y(\tau)=\sum_{i=1}^{k}\beta_i\left(\frac{a}{\lambda_i}\cos a\tau-\sin a\tau\right)E_i-\sum_{i=k+1}^{n-1}\beta_i\sin a\tau E_i$$

and

$$Y(\tau)=\sum_{i=1}^{k}\beta_i\left(-\frac{\lambda_i}{a}\sin a\tau+\cos a\tau\right)E_i-\sum_{i=k+1}^{n-1}\alpha_i\sin a\tau\,E_i.$$

For finding the location of focal point of L along γ we consider the following su
form of L-Jacobi field from the above structure:

$$Y(\tau)=\beta\left(\frac{a}{\lambda}\cos a\tau-\sin a\tau\right)E(\tau)$$

$$Y(\tau) = \beta\left(-\frac{\lambda}{a}\sin a\tau + \cos a\tau \right) E(\tau).$$

Let $\gamma(\tau_0)$ is a focal point of L along γ. Then $Y(\tau_0) = 0$, i.e.

$$\beta\left(\frac{a}{\lambda}\cos a\tau_0 - \sin a\tau_0\right) = 0,$$

$$\beta\left(-\frac{\lambda}{a}\sin a\tau_0 + \cos a\tau_0 \right) = 0.$$

Consequently

$$\cot a\tau_0 = \lambda/a$$

in both the forms. Let τ_1 be the smallest positive solution of the above equation. Then the first focal point to L along γ occurs at $\gamma(\tau_1)$. Let $\tau_1, \tau_2, \ldots, \tau_k$ are the solutions of the above equation s.t. $0 < \tau_1 < \tau_2 < \cdots < \tau_k < \beta$. Since $\lambda_1, \ldots, \lambda_k$ are eigenvalues of the symmetric linear transformation and therefore they define the principal curvatures of L at z, consequently

$$\rho_1 = 1/\lambda_1, \ldots, \rho_k = 1/\lambda_k$$

will be the corresponding principal radii of curvatures. Therefore k focal points will lie along γ at distances $\rho_1 = 1/a \tan a\tau_1, \ldots, \rho_k = 1/a \tan a\tau_k$.

Thus we have

THEOREM: Let M be a C^∞ n-dimensional compact naturally reductive Riemannian homogeneous space and L be a C^∞ k-dimensional submanifold embedded in M. Let $\gamma:[0, \beta] \to M$ be a normal geodesic with $\gamma(0) = z \in L, \dot{\gamma}(0) = u \in N_z L$. If

$$Y(\tau) = \sum_{i=1}^{n-1} (\alpha_i(\tau)\cos a\tau - \beta_i(\tau)\sin a\tau)\, E_i$$

$$Y(\tau) = \sum_{i=1}^{n-1} (\alpha_i(\tau)\sin a\tau + \beta_i(\tau)\cos a\tau)\, E_i$$

where $(\dot{\gamma}(\tau), E_1(\tau), \ldots, E_{n-1}(\tau))$ is an orthonormal base for $T_{\gamma(\tau)}M$, are Jacobi fields defined along γ. Then the location of k-focal points of L along γ is given by the equation

$$\frac{1}{a}\tan a\tau_i = \rho_i, \quad i = 1, \ldots, k.$$

k focal points along γ will generate k sets of focal locus in M.

4. Special form of Jacobi fields and focal points

A vector field X is called a Killing vector field if the operator $A_X = \nabla_X$ is skew-

$=(d/ds)|_{s=0}\varphi_s\circ\gamma(t)$ and for each s, $\varphi_s\circ\gamma(t)$ is a geodesic. Jacobi fields which are restrictions of Killing vector fields are called isotropic Jacobi fields. An isotropic Jacobi field Y along

$$\gamma(\tau)=\exp\tau v,\quad v\in\mathfrak{m}$$

satisfies [8], [2]

$$Y(0)=\begin{cases}0 & \text{if } X\in\mathfrak{k},\\ X & \text{if } X\in\mathfrak{m};\end{cases}$$

and

$$\nabla Y(0)=\begin{cases}[X,v] & \text{if } X\in\mathfrak{k},\\ \tfrac{1}{2}[X,v]_{\mathfrak{m}} & \text{if } X\in\mathfrak{m}.\end{cases}$$

Let

$$E=\{W\in T_{p_0}M\mid\langle v,W\rangle=0\},$$

if $X\in E\subset\mathfrak{m}$ then also

$$[X,v]_{\mathfrak{m}}\in E:\langle[X,v]_{\mathfrak{m}},v\rangle=-\langle X,[v,v]_{\mathfrak{m}}\rangle=0.$$

Thus the Jacobi fields coming from $X\in\mathfrak{m}$ can be restricted to $X\in E$ and all Jacobi fields with initial condition:

$$[Y(0),\nabla Y(0)]=(X,[\tfrac{1}{2}X,v]_{\mathfrak{m}}),\quad X\in E,$$

are isotropic. These are already half of all Jacobi fields.

To study Jacobi fields coming from $X\in\mathfrak{k}$ we examine the symmetric endomorphism

$$B(X)=-[v,[v,X]_{\mathfrak{k}}].$$

Since $B(v)=0$, B maps E into itself and let X_i,λ_i be the eigenvectors and eigenvalues of $B|E: B(X_i)=\lambda_i X_i$ and we set $Z_i=[v,X_i]_{\mathfrak{k}}\in\mathfrak{k}$. Then

$$[Z_i,v]=[[v,X_i]_{\mathfrak{k}},v]=B(X_i)=\lambda_i X_i.$$

Therefore if $\lambda_i\neq 0$, the Jacobi field X_i corresponding to $Z_i\in\mathfrak{k}$ does vanish identically since $\nabla Y_i(0)=[Z_i,v]\neq 0$.

Let $E=E_0\oplus E_1$ with E_0 the 0-eigenspace of B and E_1 the sum of the eigenspaces with $\lambda_i\neq 0$. Then the Jacobi fields with initial condition

$$[Y(0),\nabla Y(0)]=(0,X),\quad X\in E_1,$$

are isotropic Jacobi fields.

If $X\in E_0$, i.e., $B(X)=0$, then the D parallel vector field $Y(t)=d(L_{\exp_G\tau v})_{p_0}(X)$ is a Jacobi field. The initial conditions are:

$$Y(0)=X\in E_0$$

$$\nabla_v Y(0)=D_v Y(0)-\tfrac{1}{2}T[v,Y(0)]=\tfrac{1}{2}[v,X]_{\mathfrak{m}}.$$

and $E_3 = E_2^{\perp}$. Thus $E = E_1 \oplus E_2 \oplus E_3$. Define the subspaces $V_i \subset E \oplus E$ by:

$$V_1 = \{(X, \tfrac{1}{2}[X, v]_m) | X \in E_1 \oplus E_3\},$$

$$V_2 = \{(0, X) | X \in E_1\},$$

$$V_3 = \{(X, \tfrac{1}{2}[v, X]_m | X \in E_2\},$$

$$V_4 = \{(X, \tfrac{1}{2}[v, X]_m) | X \in E_3\},$$

$$V_5 = \{(Z, X + \tfrac{1}{2}[v, X]_m) | X \in E_2,$$

$$B(X) = T(X) = [X, v]_m\}.$$

It has been shown in [8] that $E \oplus E = \oplus_{i=1}^{5} V_i$ and the following theorem has been proved:

THEOREM: $E \oplus E = \oplus_{i=1}^{5} V_i$. On a naturally reductive homogeneous space, the Jacobi fields along $\gamma(\tau)$ can be written as linear combination of Jacobi fields with initial conditions in V_i.

Jacobi fields with initial conditions in V_i can be considered in several forms. First we consider a Jacobi field $Y(\tau)$ orthogonal to $\dot{\gamma}(\tau)$, corresponding to $X_1 \in V_1$, $X_2 \in V_2$ in the form

$$Y(\tau) = \alpha X_1 + \beta X_2, \quad \alpha, \beta \in R.$$

Therefore

$$Y(0) = \alpha X_1$$

and

$$Y'(0) = \tfrac{1}{2}\alpha[X_1(0), v]_m + \beta X_2(0).$$

Thus

$$Y'(0) + \sigma_{\dot{\gamma}(0)} Y(0) = \tfrac{1}{2}\alpha[X_1(0), v]_m + \beta X_2(0) + \lambda_1 \alpha X_1(0)$$

where λ_1 is the eigenvalue of $\sigma_{\dot{\gamma}}$ applied on eigenvector X_1. Now $\frac{1}{2}\alpha[X_1, v]_m$ can be decomposed tangential and normal to the submanifold L at $\gamma(0) = z$, therefore

$$Y'(0) + \sigma_{\dot{\gamma}(0)} Y(0) = \frac{\alpha}{2}[X_1, v]_m|_{T_z L} + \lambda_1 \alpha X_1$$

$$+ \beta X_2 + \frac{\alpha}{2}[X_1, v]_m|_{N_z L},$$

and the right hand side will be the element of $N_z L$ if

$$\beta X_2(0) = -\frac{\alpha}{2}[X_1(0), v]_m|_{T_z L} - \lambda_1 \alpha X_1(0).$$

(i) $\langle Y(\tau), \dot{\gamma}(\tau) \rangle = 0,$

which is satisfied by the assumption that Jacobi field is always orthogonal to $\dot{\gamma}$.

(ii) $Y(0) \in T_z L,$

which is obvious since $Y(0) = \alpha X_1^*(0) \in T_z L$.

(iii) $$Y'(0) + \sigma_{\dot{\gamma}(0)} Y(0) = \frac{\alpha}{2}[X_1(0), v]_m|_{N_z L} \in N_z L,$$

with

$$\beta X_2(0) = -\frac{\alpha}{2}[X_1(0), v]_m|_{T_z L} - \lambda_1 \alpha X_1(0).$$

Therefore the form of L-Jacobi field is

$$Y(\tau) = \alpha(1 - \lambda_1) X_1(\tau) - \frac{\alpha}{2}[X_1(\tau), v]_m|_{T_z L}.$$

Let $\gamma(\tau_0)$ be a focal point along $\gamma(\tau)$, then

$$Y(\tau_0) = 0$$

i.e.

$$(1 - \lambda_1) X_1(\tau_0) = \tfrac{1}{2}[X_1(\tau_0), v]_m|_{T_z L}.$$

Thus we have

THEOREM 4.1: Let M be a C^∞ homogeneous compact naturally reductive Riemannian space and L be a C^∞ submanifold embedded in M. Let $\gamma: [0, \beta] \to M$ be a normal geodesic with $\gamma(0) = z \in L$, $\dot{\gamma}(0) = u \in N_z L$. If a Jacobi field $Y(\tau)$ defined orthogonally along $\gamma(\tau)$ has the form $Y(\tau) = \alpha X_1(\tau) + \beta X_2(\tau), \alpha, \beta \in R$, with initial conditions in $V_i (i = 1, 2)$. Then a point $\gamma(\tau_0)$ along $\gamma(\tau)$ is a focal point if the following condition is satisfied:

$$(1 - \lambda_1) X_1(\tau_0) = \tfrac{1}{2}[X_1(\tau_0), v]_m|_{T_z L}.$$

From the definition of V_1, V_2, V_3, it is evident that if the Jacobi field $Y(\tau)$ defined along geodesic $\gamma(\tau)$ corresponding to $X_2 \in V_2, X_3 \in V_3$ has the form

$$Y(\tau) = \alpha_1 X_2(\tau) + \beta_1 X_3(\tau), \quad \alpha_1, \beta_1 \in R,$$

then a point $\gamma(\tau_0')$ is a focal point along γ if

$$(1 - \lambda_3) X_3(\tau_0') = \tfrac{1}{2}[v, X_3(\tau_0')]_m|_{T_z L}.$$

Consider the Jacobi field $Y(\tau)$ corresponding to $X_1 \in V_1$, $X_3 \in V_3$ in the form

$$Y(\tau) = \alpha_2 X_1(\tau) + \beta_2 X_3(\tau), \quad \alpha_2, \beta_2 \in R.$$

$$Y(\tau)=\frac{1}{\lambda_3}\Bigg[(\lambda_3-\lambda_1)\alpha_2 X_1(\tau)-\Bigg\{\frac{\alpha_2}{2}[X_1(\tau),v]_m + \frac{\beta_2}{2}[v,X_3(\tau)]_m\Bigg\}\Bigg|_{T_zL}\Bigg],$$

where λ_1, λ_3 are the eigenvalues of $\sigma_{\dot\gamma(0)}$ corresponding to eigenvectors X_1 and X_3, respectively. Thus we have

THEOREM 4.2: Let M be a compact naturally reductive Riemannian homogeneous space and L be a submanifold embedded in M. If the Jacobi field defined orthogonally along normal geodesic $\gamma(\tau)$ to L has the form

$$Y(\tau)=\alpha_2 X_1(\tau)+\beta_2 X_3(\tau)$$

with initial conditions in $V_i (i=1,3)$. Then a point $\gamma(\tau_0'')$ is a focal point along $\gamma(\tau)$ if

$$(\lambda_3-\lambda_1 X_1(\tau_0'')=\frac{1}{2}\Bigg\{[X_1(\tau_0''),v]_m+\frac{\beta_2}{\alpha_2}[v,X_3(\tau_0'')]_m\Bigg\}\Bigg|_{T_zL}.$$

An analogous theorem can be obtained by considering the Jacobi fields in different forms with initial conditions in V_i.

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